Math 1 Notes

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Order of Operations

Apply "PEMDAS" left to right where

- P: Parenthesis
- E: Exponents
- M: Multiplication
- D: Division
- A: Addition
- S: Subtraction

Note that the order of multiplication/division and addition/subtraction can be switched.

Functions

A **function** is a rule that assigns one output to each input. The graph of a function passes the **vertical line test**.

A **one-to-one function** is a function where each output is assigned to one input. The graph of a one-to-one function passes the **horizontal line test**.

Memorize the graphs and some points on the graphs of following "toolkit" functions:

$$x, x^2, |x|, x^3, \sqrt{x}, \sqrt[3]{x}, \frac{1}{x}, \frac{1}{x^2}$$

These graphs are on page 12 of Section 1.1 of the textbook here: `http://www.opentextbookstore.com/precalc/`

Domain and Range

The **domain** of a function is the set of all possible inputs. Watch out for functions with

- denominators
- even roots
- piecewise functions

The **range** of a function is the set of all possible outputs.

Average Rate of Change

The **average rate of change** of $f(x)$ on the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

Notice that this is really the slope between the points $(a, f(a))$ and $(b, f(b))$.

Composition of Functions

Sometimes we wish to evaluate $f \circ g = f(g(x))$ or $g \circ f = g(f(x))$. Evaluate the inside function (if possible), then substitute into the outside function.

Transformations of Graphs

We can apply shifts, reflections, stretches/compressions to the "toolkit" functions. See the table below:

Type	Vertical	Horizontal
shifts	$y = f(x) + k$, up k, add k to y-values $y = f(x) - k$, down k, subtract k from y-values	$y = f(x - h)$, right h, add h to x-values $y = f(x + h)$, left h, subtract h from x-values
reflections	$y = -f(x)$, reflect about x-axis, negate y-values	$y = f(-x)$, reflect about y-axis, negate x-values
stretches	$y = af(x)$, multiply y-values by a	$y = f(ax)$, divide x-values by a

If there are multiple transformations, write function in the form

$$y = af(b(x - h)) + k$$

and apply transformations left to right; i.e. apply reflections and stretches first, apply shifts last.
A function is **even** if $f(-x) = f(x)$, **odd** if $f(-x) = -f(x)$, **neither** if it is neither even or odd.

Inverse Functions

We can find inverse of one-to-one functions. If $f(x)$ is a function and $f^{-1}(x)$ is the inverse function of f, then

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

To find the inverse of a function $f(x)$, we apply the following steps:

1. Set $y = f(x)$.
2. Swap x and y.
3. Solve for y.
4. Set $y = f^{-1}(x)$.
5. (Optional) Check $f(f^{-1}(x)) = x$, $f^{-1}(f(x)) = x$.

Linear Equations

The slope between two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

If we know the slope m as well as the y-int b, we can write the equation of the line using the slope-intercept formula:

$$y = mx + b$$

If we know the slope m as well as a point (x_1, y_1) on the line, we can write the equation of the line using the point-slope formula:

$$y - y_1 = m(x - x_1)$$

A vertical line is of the form $x = c$ for some constant c, it has undefined slope. A horizontal line is of the form $y = k$ for some constant k, it has slope 0.

Two lines $f(x) = m_1x + b_1, g(x) = m_2x + b_2$

- are **parallel** if $m_1 = m_2$
- are **perpendicular** if $m_1 = -\frac{1}{m_2}$
- intersect if $f(x) = g(x)$ has a solution.

Absolute Values

Let $c \geq 0$. Then we have the following:

$$\begin{aligned} |x| = c \quad &\Leftrightarrow \quad x = c \text{ or } x = -c \\ |x| < c \quad &\Leftrightarrow \quad -c < x < c \\ |x| > c \quad &\Leftrightarrow \quad x > c \text{ or } x < c \end{aligned}$$

Polynomials

A **polynomial** is of the form

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$$

The **degree** is the highest power available, a_n is the **leading coefficient**.

To graph a polynomial, we need to identify the following:

1. end behavior (depends on degree and leading coefficient)
2. local behavior (depends on zeros and their multiplicities)

Below is a summary of end behavior:

	even degree	odd degree
$a_n > 0$	$f(x) \to \infty$ as $x \to -\infty$ $f(x) \to \infty$ as $x \to \infty$	$f(x) \to -\infty$ as $x \to -\infty$ $f(x) \to \infty$ as $x \to \infty$
$a_n < 0$	$f(x) \to -\infty$ as $x \to -\infty$ $f(x) \to -\infty$ as $x \to \infty$	$f(x) \to \infty$ as $x \to -\infty$ $f(x) \to -\infty$ as $x \to \infty$

c is a **zero** of $p(x)$ if $p(c) = 0$. This is equivalent to $(x-c)$ being a factor of $p(x)$. c has **multiplicity** m if $(x-c)^m$ $(m \geq 1)$ is a factor of $p(x)$. If m is odd, then the graph of the polynomial crosses the x-axis at $x = c$. If m is even, the graph of the polynomial touches the x-axis at $x = c$.

Rational Functions

A rational function is of the form $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x), q(x)$ are polynomials. Use the following steps to graph:

1. Factor $p(x)$ and $q(x)$.
2. Vertical asymptotes and holes: If any terms in the numerator cancel out with terms in the denominator, then there is a **hole**; i.e. if in the factored form we have
$$\frac{x-c}{x-c},$$
there is a hole at $x = c$. Otherwise, the vertical asymptotes occur when $q(x) = 0$. If the multiplicity of the vertical asymptote is odd, then the end behavior is opposite; if the multiplicity is even, then the end behavior is the same.
3. Horizontal/slant asymptotes:
    (a) If the degree of $p(x)$ is smaller than the degree of $q(x)$, then we have a horizontal asymptote at $y = 0$.
    (b) If the degree of $p(x)$ is the same as the degree of $q(x)$, then the horizontal asymptote is given by
    $$y = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$$
    (c) If the degree of $p(x)$ is one more than the degree of $q(x)$, then we have slant asymptote. Use long division to find this asymptote.
4. x- and y-intercepts: The x-intercept occurs when the numerator $p(x) = 0$. If the multiplicity of the x-intercept is even, then the function touches the x-axis; if the x-intercept is odd, then the function crosses the x-axis.
5. Plot extra points if necessary.

Exponential Functions

An exponential function is of the form $y = a^x$ for $a > 0, a \neq 1$. The graph of $y = a^x$ passes through the points $(-1, \frac{1}{a}), (0, 1), (1, a)$.

Below are some properties of exponents:

$$
\begin{aligned}
a^{-n} &= \frac{1}{a^n} \\
a^m a^n &= a^{m+n} \\
\frac{a^m}{a^n} &= a^{m-n} \\
(a^m)^n &= a^{mn} \\
a^{m/n} &= (\sqrt[n]{a})^m
\end{aligned}
$$

Logarithmic Functions

$y = \log_a x$ is the inverse of a^x. The graph of $y = \log_a x$ passes through the points $(\frac{1}{a}, -1), (1, 0), (a, 1)$.

Below are some properties of exponents:

$$
\begin{aligned}
\log_a a^x &= x \\
a^{\log_a x} &= x \\
y = a^x \quad &\Leftrightarrow \quad x = \log_a y \\
\log_a(AB) &= \log_a A + \log_a B \\
\log_a\left(\frac{A}{B}\right) &= \log_a A - \log_a B \\
\log_a(A^c) &= c \log_a A
\end{aligned}
$$

Exponential growth, decay, and interest compounded continuously can all be modeled by the following equation:

$$P(t) = P_0 e^{rt}$$

where $P(t)$ is the amount at time t, P_0 is the initial amount, r is the rate (we expect a positive rate for growth, negative right for decay), and t is the time.

Newton's Law of Cooling is modeled by the equation

$$T(t) = T_s + (T_0 - T_s)e^{-kt}$$

where $T(t)$ is the temperature at time t, T_s is the surround temperature, T_0 is the initial temperature, k is the rate, and t is the time.

Circles

The equation of a circle with radius r and center (h,k) is given by the equation

$$(x-h)^2+(y-k)^2=r^2$$

Angles

We can convert between degrees and radians using the following conversions:

$$2\pi \text{ radians} = 360 \text{ degrees} \quad \text{or} \quad \pi \text{ radians} = 180 \text{ degrees}$$

The **length of an arc** s of a circle with radius r subtended by an angle θ (in radians) is given by

$$s = r\theta.$$

The **area of a sector** A of a circle with radius r subtended by an angle θ (in radians) is given by

$$A = \frac{1}{2}\theta r^2.$$

Trigonometric Functions

$\sin x$ and $\cos x$ each have amplitude 1, period 2π and midline $y=0$. The transformations

$$f(x) = a\sin(b(x-h))+k \quad \text{and} f(x) = a\cos(b(x-h))+k$$

each have amplitude $|a|$, period $\frac{2\pi}{b}$, and midline $y=k$.
$\sin x$ is an odd function, i.e. $\sin(-x) = -\sin(x)$. $\cos x$ is an even function, i.e. $\cos{(-x)} = \cos{(x)}$.

$\sin^{-1}(x)$ is the inverse function of $\sin x$ and returns an angle between $-\frac{\pi}{x}$ and $\frac{\pi}{2}$. $\cos^{-1}(x)$ is the inverse function of $\cos x$ and returns an angle between 0 and π.
Sum and Difference Identitites:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$