WEEK 2 SOLUTIONS

① (a) $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta = 2\cdot 3\cos(120^\circ) = 6\left(-\frac{1}{2}\right) = -3$

(b) $\|2\vec{v}+\vec{w}\|^2 = (2\vec{v}+\vec{w})\cdot(2\vec{v}+\vec{w}) = 4\vec{v}\cdot\vec{v} + 2\vec{v}\cdot\vec{w} + 2\vec{w}\cdot\vec{v} + \vec{w}\cdot\vec{w}$ but $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$, so

$= 4\|\vec{v}\|^2 + 4\vec{v}\cdot\vec{w} + \|\vec{w}\|^2 = 4(2)^2 + 4(-3) + (3)^2 = 16 - 12 + 9 = 13 \Rightarrow \|2\vec{v}+\vec{w}\| = \sqrt{13}$

(c) $\|2\vec{v}-3\vec{w}\|^2 = (2\vec{v}-3\vec{w})\cdot(2\vec{v}-3\vec{w}) = 4\vec{v}\cdot\vec{v} - 6\vec{v}\cdot\vec{w} - 6\vec{w}\cdot\vec{v} + 9\vec{w}\cdot\vec{w}$

$= 4\|\vec{v}\|^2 - 12\vec{v}\cdot\vec{w} + 9\|\vec{w}\|^2 = 4(2)^2 - 12(-3) + 9(3)^2 = 16 + 36 + 81 = 133$

$\Rightarrow \|2\vec{v}-3\vec{w}\| = \sqrt{133}$

② Given $\|\vec{e}\| = 1$, $\|\vec{f}\| = 1$

$\|\vec{e}+\vec{f}\| = \sqrt{\frac{3}{2}} \Rightarrow \|\vec{e}+\vec{f}\|^2 = \frac{3}{2}$. But $\|\vec{e}+\vec{f}\|^2 = (\vec{e}+\vec{f})\cdot(\vec{e}+\vec{f}) = \vec{e}\cdot\vec{e} + \vec{e}\cdot\vec{f} + \vec{f}\cdot\vec{e} + \vec{f}\cdot\vec{f}$

$= \|\vec{e}\|^2 + 2\vec{e}\cdot\vec{f} + \|\vec{f}\|^2 = 1 + 2\vec{e}\cdot\vec{f} + 1$

$= 2 + 2\vec{e}\cdot\vec{f}$

$2 + 2\vec{e}\cdot\vec{f} = \frac{3}{2}$

$2\vec{e}\cdot\vec{f} = -\frac{1}{2}$

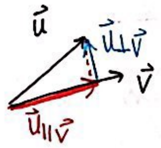
$\vec{e}\cdot\vec{f} = -\frac{1}{4}$

$\|2\vec{e}-3\vec{f}\|^2 = (2\vec{e}-3\vec{f})\cdot(2\vec{e}-3\vec{f}) = 4\vec{e}\cdot\vec{e} - 6\vec{e}\cdot\vec{f} - 6\vec{f}\cdot\vec{e} + 9\vec{f}\cdot\vec{f} = 4\|\vec{e}\|^2 - 12\vec{e}\cdot\vec{f} + 9\|\vec{f}\|^2$

$= 4(1)^2 - 12\left(-\frac{1}{4}\right) + 9(1)^2 = 4 + 3 + 9 = 16$

$\Rightarrow \|2\vec{e}-3\vec{f}\| = \sqrt{16} = 4$

③ $\vec{u} = \langle 3,5\rangle$, $\vec{v} = \langle 8,2\rangle$



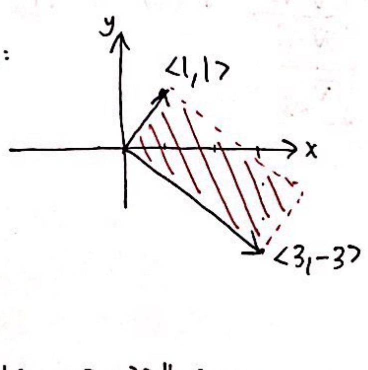
red: $\vec{u}_{\parallel\vec{v}} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \left(\frac{\langle 3,5\rangle\cdot\langle 8,2\rangle}{\langle 8,2\rangle\cdot\langle 8,2\rangle}\right)\langle 8,2\rangle = \frac{24+10}{64+4}\langle 8,2\rangle = \frac{34}{68}\langle 8,2\rangle$

$= \frac{1}{2}\langle 8,2\rangle = \langle 4,1\rangle$

blue: $\vec{u}_{\perp\vec{v}} = \vec{u} - \vec{u}_{\parallel\vec{v}} = \langle 3,5\rangle - \langle 4,1\rangle = \langle -1,4\rangle$

$\|\vec{u}_{\perp\vec{v}}\| = \sqrt{(-1)^2 + 4^2} = \sqrt{1+16} = \sqrt{17}$

④ geometrically:



By Right hand Rule, $\langle 1,1\rangle \times \langle 3,-3\rangle$ will have the thumb pointing into the page $\Rightarrow -\hat{k}$ direction

$\|\langle 1,1\rangle\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\|\langle 3,-3\rangle\| = \sqrt{9+9} = \sqrt{18}$

multiply the magnitudes together to get magnitude of the cross product $= \sqrt{2}\cdot\sqrt{18} = \sqrt{36} = 6$

Therefore $\langle 1,1\rangle \times \langle 3,-3\rangle = -6\hat{k}$

* Note: $\|\langle 1,1\rangle \times \langle 3,-3\rangle\|$ is the area of the red parallelogram above, so you could also use the area of the parallelogram to get a magnitude of 6

④ continued

algebraically: $\langle 1,1\rangle = \hat{\imath} + \hat{\jmath}$ $\Rightarrow$ $\langle 1,1\rangle \times \langle 3,-3\rangle = (\hat{\imath}+\hat{\jmath})\times(3\hat{\imath}-3\hat{\jmath})$

$\langle 3,-3\rangle = 3\hat{\imath} - 3\hat{\jmath}$

$= \cancel{3\hat{\imath}\times\hat{\imath}}^{\,0} - 3\hat{\imath}\times\hat{\jmath} + 3\hat{\jmath}\times\hat{\imath} - \cancel{3\hat{\jmath}\times\hat{\jmath}}^{\,0}$

$= -3\hat{\imath}\times\hat{\jmath} - 3\hat{\imath}\times\hat{\jmath}$ since $\hat{\imath}\times\hat{\jmath} = -\hat{\jmath}\times\hat{\imath}$

$= -6\hat{\imath}\times\hat{\jmath} = -6\hat{k}$

⑤ $\langle a,b,0\rangle \times \langle c,d,0\rangle = (a\hat{\imath}+b\hat{\jmath})\times(c\hat{\imath}+d\hat{\jmath}) = \cancel{ac\,\hat{\imath}\times\hat{\imath}}^{\,0} + ad\,\hat{\imath}\times\hat{\jmath} + bc\,\hat{\jmath}\times\hat{\imath} + \cancel{bd\,\hat{\jmath}\times\hat{\jmath}}^{\,0}$

$= ad\,\hat{\imath}\times\hat{\jmath} - bc\,\hat{\imath}\times\hat{\jmath} = (ad-bc)\,\hat{\imath}\times\hat{\jmath} = \boxed{(ad-bc)\,\hat{k}}$

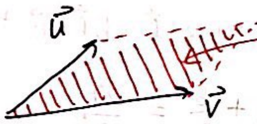
Using determinant:

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = \hat{\imath}\begin{vmatrix} b & 0 \\ d & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$= (\cancel{b\cdot 0 - d\cdot 0}^{\,0})\,\hat{\imath} - (\cancel{a\cdot 0 - c\cdot 0}^{\,0})\,\hat{\jmath} + (ad-bc)\,\hat{k}$

$= \boxed{(ad-bc)\,\hat{k}}$ same as above

Areas: The magnitude of the cross product is the area of the parallelogram spanned by the two vectors, e.g.



this area is $\|\vec{u}\times\vec{v}\|$

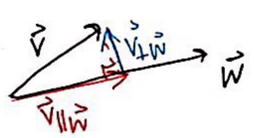
⑥ $\hat{\imath}\times\hat{k} = -\hat{\jmath}$

⑦ $\hat{k}\times\hat{\imath} = \hat{\jmath}$

⑧ $(d\hat{\imath}+f\hat{k})\times(x\hat{\imath}+z\hat{k}) = \cancel{dx\,\hat{\imath}\times\hat{\imath}}^{\,0} + dz\,\hat{\imath}\times\hat{k} + fx\,\hat{k}\times\hat{\imath} + \cancel{fz\,\hat{k}\times\hat{k}}^{\,0}$

$= dz(-\hat{\jmath}) + fx\,\hat{\jmath} = -dz\,\hat{\jmath} + fx\,\hat{\jmath} = \boxed{(fx-dz)\,\hat{\jmath}}$

⑨



If $\vec{w}$ is nonzero, then we can compute the projection $\vec{v}_{\parallel\vec{w}} = \left(\dfrac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$

Then we can compute $\vec{v}_{\perp\vec{w}} = \vec{v} - \vec{v}_{\parallel\vec{w}}$

⑩ $\vec{v}\times\vec{w} = (\vec{v}_{\parallel\vec{w}} + \vec{v}_{\perp\vec{w}})\times\vec{w} = \vec{v}_{\parallel\vec{w}}\times\vec{w} + \vec{v}_{\perp\vec{w}}\times\vec{w} = \left(\dfrac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\cancel{\vec{w}\times\vec{w}}^{\,0} + \vec{v}_{\perp\vec{w}}\times\vec{w}$

$\uparrow$

plug in decomposition form

$= \boxed{\vec{v}_{\perp\vec{w}}\times\vec{w}}$