~~Week~~ 5 Solutions
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① $\vec{r}(t) = \langle 3\cos t, 3\sin t, 0\rangle$

$\vec{r}'(t) = \langle -3\sin t, 3\cos t, 0\rangle$ $\Rightarrow$ $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \langle -\sin t, \cos t, 0\rangle$

$\|\vec{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 0^2} = 3$

$\vec{T}'(t) = \langle -\cos t, -\sin t, 0\rangle$ $\Rightarrow$ $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \langle -\cos t, \sin t, 0\rangle$

$\|\vec{T}'(t)\| = \sqrt{(-\cos t)^2 + (-\sin t)^2 + 0^2} = 1$

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \hat{\imath}\begin{vmatrix} \cos t & 0 \\ -\sin t & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -\sin t & 0 \\ -\cos t & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}$$

$$= \langle 0, 0, +\sin^2 t + \cos^2 t\rangle = \boxed{\langle 0, 0, 1\rangle}$$

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② $\vec{r}(t) = \langle 3\sin t, 3\cos t, 0\rangle$

$\vec{r}'(t) = \langle 3\cos t, -3\sin t, 0\rangle$ $\Rightarrow$ $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \langle \cos t, -\sin t, 0\rangle$

$\|\vec{r}'(t)\| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + 0^2} = 3$

$\vec{T}'(t) = \langle -\sin t, -\cos t, 0\rangle$ $\Rightarrow$ $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \langle -\sin t, -\cos t, 0\rangle$

$\|\vec{T}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 0^2} = 1$

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos t & -\sin t & 0 \\ -\sin t & -\cos t & 0 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -\sin t & 0 \\ -\cos t & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \cos t & 0 \\ -\sin t & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{vmatrix}$$

$$= \langle 0, 0, -\cos^2 t - \sin^2 t\rangle = \boxed{\langle 0, 0, -1\rangle}$$

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③ $\vec{r}(s) = \langle 3\cos(\frac{s}{5}), 3\sin(\frac{s}{5}), \frac{4s}{5}\rangle$

$\vec{r}'(s) = \langle \frac{3}{5}\cos(\frac{s}{5}), \frac{3}{5}\sin(\frac{s}{5}), \frac{4}{5}\rangle$ $\Rightarrow$ $\vec{T}(s) = \dfrac{\vec{r}'(s)}{\|\vec{r}'(s)\|} = \langle \frac{3}{5}\cos(\frac{s}{5}), \frac{3}{5}\sin(\frac{s}{5}), \frac{4}{5}\rangle$

$\|\vec{r}'(s)\| = \sqrt{\left(\frac{3}{5}\cos(\frac{s}{5})\right)^2 + \left(\frac{3}{5}\sin(\frac{s}{5})\right)^2 + \left(\frac{4}{5}\right)^2} = 1$

(i.e. parametrized w.r.t. arc length)

INCORRECT

Incorrect

③ continued

$$\frac{d\vec{T}}{ds} = \vec{T}'(s) = \left\langle \frac{-3}{25}\sin\left(\frac{s}{5}\right), \frac{3}{25}\cos\left(\frac{s}{5}\right), 0 \right\rangle \Rightarrow \vec{N}(s) = \frac{\vec{T}'(s)}{\|\vec{T}'(s)\|} = \left\langle -\sin\left(\frac{s}{5}\right), \cos\left(\frac{s}{5}\right), 0 \right\rangle$$

$$\|\vec{T}'(s)\| = \sqrt{\left(\frac{-3}{25}\sin\left(\frac{s}{5}\right)\right)^2 + \left(\frac{3}{25}\cos\left(\frac{s}{5}\right)\right)^2 + 0^2} = \frac{3}{25}$$

$\uparrow$

* note: this is the curvature!

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5}\cos\left(\frac{s}{5}\right) & \frac{3}{5}\sin\left(\frac{s}{5}\right) & \frac{4}{5} \\ -\sin\left(\frac{s}{5}\right) & \cos\left(\frac{s}{5}\right) & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} \frac{3}{5}\sin\left(\frac{s}{5}\right) & \frac{4}{5} \\ \cos\left(\frac{s}{5}\right) & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{3}{5}\cos\left(\frac{s}{5}\right) & \frac{4}{5} \\ -\sin\left(\frac{s}{5}\right) & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{3}{5}\cos\left(\frac{s}{5}\right) & \frac{3}{5}\sin\left(\frac{s}{5}\right) \\ -\sin\left(\frac{s}{5}\right) & \cos\left(\frac{s}{5}\right) \end{vmatrix}$$

$$= \left\langle -\frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5}\sin\left(\frac{s}{5}\right), \frac{3}{5} \right\rangle$$

INCORRECT

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④ $$\frac{d\vec{B}}{ds} = \left\langle \frac{4}{25}\sin\left(\frac{s}{5}\right), \frac{4}{25}\cos\left(\frac{s}{5}\right), 0 \right\rangle = -\frac{4}{25}\vec{N}(s)$$

$$\tau(s) = \frac{4}{25}$$

③ $$\vec{r}(s) = \langle 3\cos(s/5), 3\sin(s/5), 4s/5 \rangle$$

$$\vec{r}'(s) = \left\langle -\frac{3}{5}\sin(s/5), \frac{3}{5}\cos(s/5), \frac{4}{5} \right\rangle$$

$$\|\vec{r}'(s)\| = \sqrt{\frac{9}{25}\sin^2\left(\frac{s}{5}\right) + \frac{9}{25}\cos^2\left(\frac{s}{5}\right) + \frac{16}{25}} = 1 \Rightarrow$$ parametrized w.r.t. arclength!

$$\vec{T}(s) = \frac{\vec{r}'(s)}{\|\vec{r}'(s)\|} = \left\langle -\frac{3}{5}\sin\left(\frac{s}{5}\right), \frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5} \right\rangle$$

$$\frac{d\vec{T}}{ds} = \left\langle -\frac{3}{25}\cos\left(\frac{s}{5}\right), \frac{-3}{25}\sin\left(\frac{s}{5}\right), 0 \right\rangle$$

$$\kappa = \left\|\frac{d\vec{T}}{ds}\right\| = \sqrt{\frac{9}{(25)^2}\cos^2\left(\frac{s}{5}\right) + \frac{9}{(25)^2}\sin^2\left(\frac{s}{5}\right) + 0} = \frac{3}{25}$$

$$\vec{N}(s) = \frac{\frac{d\vec{T}}{ds}}{\left\|\frac{d\vec{T}}{ds}\right\|} = \frac{\left\langle -\frac{3}{25}\cos\left(\frac{s}{5}\right), \frac{-3}{25}\sin\left(\frac{s}{5}\right), 0 \right\rangle}{\frac{3}{25}} = \left\langle -\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0 \right\rangle$$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{3}{5}\sin\left(\frac{s}{5}\right) & \frac{3}{5}\cos\left(\frac{s}{5}\right) & \frac{4}{5} \\ -\cos\left(\frac{s}{5}\right) & -\sin\left(\frac{s}{5}\right) & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} \frac{3}{5}\cos(s/5) & 4/5 \\ -\sin(s/5) & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} -\frac{3}{5}\sin\left(\frac{s}{5}\right) & 4/5 \\ -\cos(s/5) & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} -\frac{3}{5}\sin(s/5) & \frac{3}{5}\cos(s/5) \\ -\cos(s/5) & -\sin(s/5) \end{vmatrix}$$

$$= \left\langle \frac{4}{5}\sin\left(\frac{s}{5}\right), -\frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\sin^2(s/5) + \frac{3}{5}\cos^2\left(\frac{s}{5}\right) \right\rangle$$

$$= \boxed{\left\langle \frac{4}{5}\sin\left(\frac{s}{5}\right), \frac{-4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5} \right\rangle}$$

④ $\frac{d\vec{B}}{ds} = \langle \frac{4}{25}\cos\left(\frac{s}{5}\right), +\frac{4}{25}\sin\left(\frac{s}{25}\right), 0 \rangle = \underbrace{\frac{-4}{25}}_{\tau} \vec{N}(s).$