KEY

Directional Derivatives

In all that follows, let $\vec{u} = \langle h, k \rangle$ be a unit vector. Let $f(x, y)$ be a function that is continuous near (a, b) and such that the partials exist and are continuous near (a, b). These problems unpack slightly the notion that the directional derivative can be viewed as the slope of a tangent line.

1. Find the equation of the plane that contains the vectors $\vec{k}$ and $\vec{u}$ and passes through the point (a, b).

need a point and a normal

$\vec{u}$ in 3D is $\langle h, k, 0\rangle$ ($\vec{u} = \langle h, k\rangle$ is in the xy plane)

$\vec{k} = \langle 0, 0, 1\rangle$

$$\vec{n} = \vec{u} \times \hat{k} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ h & k & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} k & 0 \\ 0 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} h & 0 \\ 0 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} h & k \\ 0 & 0 \end{vmatrix}$$

$= \langle k, -h, 0\rangle$

Tangent plane: $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$

$\langle k, -h, 0\rangle \cdot \langle x-a, y-b, z-z_0\rangle = 0 \Rightarrow \boxed{k(x-a) - h(y-b) = 0}$

2. Write down equations that describe the intersection of your plane with the surface $z = f(x, y)$.

3. Give a parametric equation for the line parallel to $\vec{u}$ and passing through (a, b). Use this to find a parameterization of the curve given by the intersection of your plane with the surface.

② $\begin{cases} z = f(x, y) \\ k(x-a) - h(y-b) = 0 \end{cases}$

③ line: $\vec{s}(t) = \underset{\text{point}}{\vec{s_0}} + t\underset{\text{direction}}{\vec{v}} \Rightarrow \boxed{\vec{s}(t) = \langle a, b\rangle + t\underbrace{\langle h, k\rangle}_{\vec{u}} = \langle ht + a, kt + b\rangle}$

substitute into ②: $z = f(\vec{s}(t))$

$k(ht + a - a) - h(kt + b - b) = 0$

$\Rightarrow kht - hkt = 0 \Rightarrow 0 = 0.$

So the parametrization is given by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

$\boxed{\vec{r}(t) = \langle ht + a, kt + b, f(\vec{s}(t))\rangle}$

4. Find the tangent vector to the curve, using the parameterization you worked out in the previous part.

Need to find $\vec{r}\,'(t)$.

$\vec{r}(t) = \langle ht+a,\ kt+b,\ f(\vec{s}(t)) \rangle$

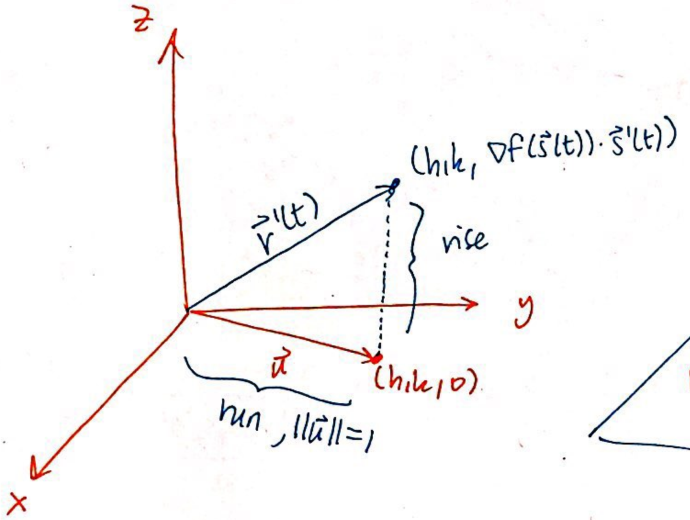
$$\boxed{\vec{r}\,'(t) = \langle \underbrace{h,\ k}_{\vec{u}},\ \underbrace{\nabla f(\vec{s}(t))\cdot \vec{s}\,'(t)}_{\text{chain rule}} \rangle}$$

Observe: $\nabla f(\vec{s}(t))\cdot\vec{s}\,'(t) = \langle f_x(\vec{s}(t)),\ f_y(\vec{s}(t)) \rangle \cdot \langle h, k \rangle$

$= h f_x(\vec{s}(t)) + k f_y(\vec{s}(t))$.

5. What do you notice about the first two coordinates? Explain how the "unit vector" condition on $\vec{u}$ gives us an interpretation of $D_{\vec{u}}f(a,b)$ as the slope of this tangent line.

First two coordinates are $\langle h, k \rangle = \vec{u}$.



$\text{slope} = \dfrac{\text{rise}}{\text{run}} = \dfrac{\nabla f(\vec{s}(t))\cdot\vec{s}\,'(t)}{1}$

since $\|\vec{u}\| = 1$

$= h f_x(\vec{s}(t)) + k f_y(\vec{s}(t))$

But $D_u f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \underbrace{\langle h, k \rangle}_{\vec{u}}$

$= h f_x(a,b) + k f_y(a,b)$

same when $t = 0$!

Scanned with CamScanner