① $f = F(r)$ where $r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$, $\vec{e_r} = \frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$

Want $\nabla f = \langle f_x, f_y, f_z\rangle$

For f_x, f is a function of r, which is a function of x: $f_x = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} = \frac{dF}{dr}\cdot\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x$

$= \frac{dF}{dr}\frac{x}{\sqrt{x^2+y^2+z^2}}$

similarly, $f_y = \frac{\partial F}{\partial r}\frac{\partial r}{\partial y} = \frac{dF}{dr}\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2y = \frac{dF}{dr}\frac{y}{\sqrt{x^2+y^2+z^2}}$

$f_z = \frac{\partial F}{\partial r}\frac{\partial r}{\partial z} = \frac{dF}{dr}\cdot\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2z = \frac{dF}{dr}\frac{z}{\sqrt{x^2+y^2+z^2}}$

Then $\nabla f = \langle f_x, f_y, f_z\rangle = \langle \frac{x}{\sqrt{x^2+y^2+z^2}}F'(r), \frac{y}{\sqrt{x^2+y^2+z^2}}F'(r), \frac{z}{\sqrt{x^2+y^2+z^2}}F'(r)\rangle$

$= F'(r)\underbrace{\frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}}_{\text{This is } \vec{e_r}} = F'(r)\vec{e_r}$ ✓

Want $z - z_0 = \frac{\partial z}{\partial x}(x-x_0) + \frac{\partial z}{\partial y}(y-y_0)$

② One way Define $f(x,y,z) = x^2y+y^2z+xz^2-3$ $\Rightarrow$ $f_x = 2xy+z^2$, $f_y = x^2+2yz$, $f_z = y^2+2xz$

at $(1,1,1)$: $f_x = 3$, $f_y = 3$, $f_z = 3$

use $\frac{\partial z}{\partial x} = \frac{-f_x}{f_z}$, $\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}$

$= \frac{-3}{3} = -1$, $= \frac{-3}{3} = -1$

$\Rightarrow$ $\boxed{z-1 = -1(x-1)-1(y-1)}$ or $x+y+z=3$

Another way $z = z(x,y)$, do implicit differentiation like in single-variable calculus (i.e. when we take the derivative of z, multiply by $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$)

$x^2y+y^2z+xz^2=3$. Take derivative w/ respect to x: $2xy + y^2\frac{\partial z}{\partial x} + \overbrace{1\cdot z^2 + x\cdot 2z\frac{\partial z}{\partial x}}^{\text{product rule b.c. } z \text{ is function of } x} = 0$

solve for $\frac{\partial z}{\partial x}$: $\frac{\partial z}{\partial x}(y^2+x2z) = -z^2-2xy$

$\frac{\partial z}{\partial x} = \frac{-z^2-2xy}{y^2+x^2z} = \frac{-1-2}{1+2} = -1$

Take derivative w.r.t. to y: $x^2 + \underbrace{2yz + y^2\frac{\partial z}{\partial x}}_{\text{product rule b.c. } z \text{ is function of } y} + 2xz\frac{\partial z}{\partial y} = 0 \Rightarrow$ $\frac{\partial z}{\partial x}(y^2+2xz) = -x^2-2yz$

$\frac{\partial z}{\partial x} = \frac{-x^2-2yz}{y^2+2xz} = \frac{-3}{3} = -1$

$\Rightarrow$ $\boxed{z-1 = -1(x-1)-1(y-1)}$

*Note: can also find tangent plane w/o using implicit diff., this may be a good way to check your answer.

Define $f(x,y,z) = x^2y+y^2z+xz^2-3$. $\vec{n} = \nabla f = \langle 2xy+z^2, 1+2yz, y^2+2xz\rangle = \langle 3,3,3\rangle$

Then $\vec{n}\cdot\langle x-1, y-1, z-1\rangle = 0 \Rightarrow \langle 3,3,3\rangle\cdot\langle x-1,y-1,z-1\rangle = 0 \Rightarrow 3(x-1)+3(y-1)+3(z-1) = 0$

$\Rightarrow$ $\boxed{x+y+z=3}$, same as above!

(3) f is function of x and y, x and y are each functions of s and t.

$x = s + t$ $\quad$ $y = s - t$

$\frac{\partial x}{\partial s} = 1$ $\quad$ $\frac{\partial y}{\partial s} = 1$

$\frac{\partial x}{\partial t} = 1$ $\quad$ $\frac{\partial y}{\partial t} = -1$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$$

Then $\left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial f}{\partial t}\right) = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2$ ✓

(4) one way plug directly in ⇒ $f(r,\theta) = r\cos\theta\cos^2\theta - r^2 = r\cos^3\theta - r^2 = r(\cos\theta)^3 - r^2$

$$\frac{\partial f}{\partial \theta} = 3r\cos^2\theta(-\sin\theta) = -3r\sin\theta\cos^2\theta$$

Another way Chain rule

$f(x,y,z) = xy - z^2$

$f_x = y$ $\quad$ $f_z = -2z$

$f_y = x$

$x = r\cos\theta$ $\quad$ $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = \cos^2\theta$ $\quad$ $\frac{\partial y}{\partial \theta} = 2\cos\theta(-\sin\theta)$

$z = r$ $\quad$ $\frac{\partial z}{\partial \theta} = 0$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}$$

$$= y(-r\sin\theta) + x(-2\sin\theta\cos\theta) + \cancel{(-2z)(0)}^{\,0}$$

$$= -r\sin\theta(\cos^2\theta) - 2\sin\theta\cos\theta(r\cos\theta)$$

$$= -r\sin\theta\cos^2\theta - 2r\sin\theta\cos^2\theta$$

$$= -3r\sin\theta\cos^2\theta$$ ✓ same as above!