Math 31B Notes

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Exponential and Logarithmic Functions

Some important exponent and logarithmic laws:

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn} \qquad a^{-1} = \frac{1}{a}$$

$$\log_b(mn) = \log_b m + \log_b n \qquad \log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n \qquad \log_b m^n = n\log_b m$$

$$\log_b m = \frac{\log_a m}{\log_a b} \qquad \log_b x = y \Leftrightarrow b^y = x \qquad \log_b(b^x) = x \qquad b^{\log_b x} = x$$

Derivatives:

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

To take the derivative of a logarithmic function with a different base, use the change of base formula:

$$\frac{d}{dx}\log_b x = \frac{d}{dx}\frac{\ln x}{\ln b} = \frac{1}{x\ln b}$$

L'Hôpital's Rule

If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is **indeterminate** and $f(x), g(x)$ are differentiable near a, then

$$\lim_{x\to a}\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

The following are indeterminate forms:

$$\frac{0}{0}, \quad \pm\frac{\infty}{\infty}, \quad 0\cdot\infty, \quad \infty-\infty, \quad 0^0, \quad 1^\infty, \quad \infty^0$$

Sequences

A **sequence** is a list of numbers:

$$a_1, a_2, \ldots$$

We can also think of sequences as a function $f(n)$ on the natural numbers.

A sequence **converges** to a finite limit L if

$$\lim_{n\to\infty} a_n = L.$$

If the sequence doesn't converge (e.g. the limit doesn't exist or is $\pm\infty$), then the sequence **diverges**. The following theorems are useful to determine whether a sequence converges or diverges.

- If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, then $\lim_{n\to\infty} a_n = L$. (Use this if you want to use L'Hôpital's Rule.)
- Squeeze Theorem: If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, the $\lim_{n\to\infty} b_n = L$. (Use this if you have sine or cosine terms in the sequence.)
- If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. (Use this if you have an alternating sequence.)
- If $\lim_{n\to\infty} a_n = L$ and f is a continous function at L, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

(Use this theorem if you want to move the limit inside another function.)

Series

A **series** $\displaystyle\sum_{n=1}^{\infty} a_n$ is the sum of the terms of the sequence a_n:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$$

The partial sum of a series is $s_N = \displaystyle\sum_{n=1}^{N} a_n$. A series **converges** if the limit of its partial sums converges, otherwise it diverges. We have several tests to help us determine if a series converges.

Divergence Test

An important fact about convergent series is the following: If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$. This motivates the Divergence Test.

Theorem (Divergence Test)**.** *If* $\displaystyle\lim_{n\to\infty} a_n \neq 0$, *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges.*

What happens if $\displaystyle\lim_{n\to\infty} a_n = 0$? We need to use a different test.

Geometric Series

A geometric series is of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ...$$

is convergent if $|r| < 1$, divergent if $|r| \geq 1$. If $|r| < 1$, then

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

Telescopic Series

A telescope series is a series in which several of the terms cancel out. Write out a general partial sum and then take the limit.

Integral Test and p-Series

Theorem (Integral Test)**.** *Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$.*

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem (p-Series Test)**.** *The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.*

Comparison Tests

Theorem (Direct Comparison)**.** *Suppose $a_n, b_n \geq 0$.*

(i) If $a_n \leq b_n$ and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.

(ii) If $a_n \geq b_n$ and $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

Theorem (Limit Comparison)**.** *Suppose $a_n, b_n \geq 0$.*

(i) If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c$ where $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

(ii) If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$ where $0 < c < \infty$, then $\sum a_n$ converges if $\sum b_n$ converges.

(iii) If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ where $0 < c < \infty$, then $\sum a_n$ diverges if $\sum b_n$ diverges.

Alternating Series Test

Theorem (Alternating Series Test)**.** *If*

(i) $\lim_{n\to\infty} b_n = 0$*, and*

(ii) b_n *is a decreasing sequence,*

then $\sum_{n=1}^{\infty}(-1)^n b_n$ *converges.*

To show a sequence a_n is decreasing, set $f(n) = a_n$ show that $f'(x) < 0$.

Absolute Convergence, Ratio Test, and Root Test

A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent. $\sum a_n$ is **conditionally convergent** if it is convergent but $\sum |a_n|$ is divergent.

Theorem. *If* $\sum |a_n|$ *converges then* $\sum a_n$ *converges.*

Theorem (Ratio Test)**.** *Let* $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

(i) If $L < 1$*, then* $\sum a_n$ *is absolutely convergent*

(ii) If $L > 1$*, then* $\sum a_n$ *diverges.*

(iii) If $L = 1$*, then the test is inconclusive. Use a different test.*

Theorem (Root Test)**.** *Let* $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

(i) If $L < 1$*, then* $\sum a_n$ *is absolutely convergent*

(ii) If $L > 1$*, then* $\sum a_n$ *diverges.*

(iii) If $L = 1$*, then the test is inconclusive. Use a different test.*

Power Series

A **power series** about a is given by

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

To find the radius of convergence R of a power series, use the Ratio Test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

To find the interval of convergence, evalute $|x-a| < R$ at the endpoints.
Using geometric series, we can find the formula of $f(x) = \frac{1}{1-x}, |x| < 1$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Taylor Series

The **Taylor series** representation of $f(x)$ about a is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The **Maclaurin series** if the Taylor series with $a = 0$. The following are some common Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Taylor Polynomial

The n-th degree Taylor Polynomial $T_n(x)$ of $f(x)$ about $x = a$ is given by

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The error of the Taylor Polynomial T_n is given by

$$|f(x) - T_n(x)| = \frac{K|x-a|^{n+1}}{(n+1)!} \quad \text{where} \quad K = \max_{[a,x]} |f^{(n+1)}(x)|$$

Inverse Functions

$f^{-1}(x)$ is the inverse function of $f(x)$ if $f^{-1}(f(x)) = x, f(f^{-1}(x)) = x$.

If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Integration by Parts

The formula for integration by parts is given by

$$\int u dv = uv - \int v du$$

A helpful mneumonic for choosing u is "LIATE":

Logarithms

Inverse Trig

Algebraic

Trig

Exponential

Inverse Trigonometric Functions

Below are some important derivatives and integrals:

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \quad \Leftrightarrow \quad \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$$
$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2} \quad \Leftrightarrow \quad \int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$$

Partial Fraction Decomposition

A rational function is of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x), q(x) \neq 0$ are polynomials. To evaluate the integral $\int \frac{p(x)}{q(x)}dx$ we need to use partial fraction decomposition. Use the following steps:

1. If $\deg p(x) \geq \deg q(x)$ then use long division.
2. Factor $q(x)$.
3. Write out the partial fraction decomposition using the factors of $q(x)$ in the cases below:
   (i) If $q(x)$ has distinct linear factors, e.g. $q(x) = (a_1x - b_1)(a_2x - b_2)\cdots(a_nx_n - b_n)$, then
   $$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x - b_1} + \frac{A_2}{a_2x - b_2} + \cdots + \frac{A_n}{a_nx - b_n}$$
   where $A_1, ..., A_n$ are constants.
   (ii) If $q(x)$ has distinct quadratic factors, e.g. $q(x) = (a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_nx^2 + b_nx + c_n)$, then
   $$\frac{p(x)}{q(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_nx + B_n}{a_nx^2 + b_nx + c_n}$$
   where $A_1, ..., A_n, B_1, ..., B_n$ are constants.
   (iii) If $q(x)$ has repeated linear factors, e.g. $q(x) = (ax - b)^n$, then
   $$\frac{p(x)}{q(x)} = \frac{A_1}{ax - b} + \frac{A_2}{(ax - b)^2} + \cdots + \frac{A_n}{(ax - b)^n}$$
   where $A_1, ..., A_n$ are constants.

(iv) If $q(x)$ has repeated quadratic factors, e.g. $q(x) = (ax^2 + bx + c)^n$, then

$$\frac{p(x)}{q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

where $A_1, ..., A_n, B_1, ..., B_n$ are constants.

(v) $q(x)$ may be a mix of cases (i) - (iv) $-$ how fun!

4. Solve for the constants in the numerators of the partial fraction decomposition.
5. Integrate $\int \frac{p(x)}{q(x)} dx$ using the partial fraction decomposition. Your answer will most likely have inverse tangent and/or natural log terms.

Note: You do not need to memorize the exact forms above. If you have a linear factor, a constant goes on the numerator. If you have a quadratic factor, a linear term like $Ax + B$ goes on the numerator.

Improper Integrals

There are two types of improper integrals:

1. Infinite Integrals
   (a) Rewrite $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)$.
   (b) Rewrite $\int_{-\infty}^a f(x)dx = \lim_{b\to\infty} \int_b^a f(x)$.
   (c) Rewrite $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$, then apply (a) and (b).
2. Discontinuous integrals
   (a) If $f(x)$ is discontinuous at b, then rewrite $\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$.
   (b) If $f(x)$ is discontinuous at a, then rewrite $\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$.
   (c) If $f(x)$ is discontinuous at c where $a < c < b$, then first rewrite $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, then apply (a) and (b).

If the limit exists (not infinite), then the integral is **convergent**. If the limit does not exist, the integral is **divergent**.