Math 32B Notes

Written by Victoria Kala
vtkala@math.ucla.edu
Last updated March 12, 2019

Double Integrals (Cartesian Coordinates)

When integrating over rectangles, we can apply Fubini's Theorem for integrable functions:

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

For general regions we can apply the following steps:

1. Sketch the region of integration.
2. Write the bounds of one variable in terms of constants, the other variable in terms of functions:

$$\begin{array}{ccc} a \le x \le b & \text{and} & f(x) \le y \le g(x) \\ & \text{or} & \\ c \le y \le d & \text{and} & h(y) \le x \le k(y) \end{array}$$

We can find the area over a region D, evaluate

$$A = \iint_D 1dA$$

where dA is an area element.

Polar Coordinates

Use polar coordinates whenever integrating over circular regions. Use the following change of coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad dA = rdrd\theta$$

Observe $x^2 + y^2 = r^2$.

Triple Integrals (Cartesian Coordinates)

To find the volume of a solid E, evaluate

$$V = \iiint_E 1dV$$

where dV is a volume element.

For general regions we can apply the following steps:

1. Sketch the region of integration.
2. Write the bounds of the first variable in terms of constants, the second variable in terms of functions of the first variable, the third variable in terms of functions of the first and second variables, for example

$$a \leq x \leq b, \qquad f(x) \leq y \leq g(x), \qquad h(x,y) \leq z \leq k(x,y).$$

Cylindrical Coordinates

Changing to cylindrical coordinates is most useful when integrating over parts of circular cylinders, paraboloids, or flat cones (cone beneath a plane). Use the following change of coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \qquad dV = r dz dr d\theta$$

Observe $x^2 + y^2 = r^2$.

Spherical Coordinates

Changing to spherical coordinates is most useful when integrating over parts of spheres or "rounded" cones (cone inside a sphere). We introduce a third parameter ϕ, this is the azimuthal angle measured from the positive z-axis to the negative z-axis.

$$x = \rho\cos\theta\sin\phi \qquad y = \rho\sin\theta\sin\phi \qquad z = \rho\cos\phi \qquad dV = \rho^2 \sin\phi d\rho d\phi d\theta$$

Observe $x^2 + y^2 + z^2 = \rho^2$.

Change of Variables

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y) dA = \iint_S f(x(u,v), y(u,v))|J| du dv$$

J is the Jacobian given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Vector Fields

Let $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$, $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$. The **divergence** of $\mathbf{F}$ is given by

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The **curl** of **F** is given by

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

A vector field **F** is called a **conservative vector field** if there exists a **potential function** f such that $\nabla f = \mathbf{F}$. In 2 dimensions, $\mathbf{F} = \langle F_1, F_2 \rangle$ is conservative in a simply-connected region (i.e. no holes) if

$$\frac{\partial F_2}{\partial y} = \frac{\partial F_1}{\partial x}.$$

In 3 dimensions, **F** is conservative if curl $\mathbf{F} = \mathbf{0}$.

Line Integrals

If f is a function defined along a smooth curve C defined by the parametrization $\mathbf{r}(t)$, $a \leq t \leq b$, then

$$\int_C f(x,y)ds = \int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)||dt$$

If **F** is a continuous vector field defined on a smooth curve C defined by the parametrization $\mathbf{r}(t)$, $a \leq t \leq b$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

This integral is also known as "work" or "circulation". Another form of a line integral is the following:

$$\int_C Pdx + Qdy.$$

If given a graph of a vector field **F** and a path C, we can determine the sign of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$:

- If the path travels against the vector field, then the integral is negative.
- If the path travels along the vector field, then the integral is positive.
- If the path is perpendicular to the path at all times, then the integral is zero.

Theorem (Fundamental Theorem of Line Integrals)**.** *Let C be a smooth curve given by the vector function $\mathbf{r}(t), a \leq t \leq b$. Let f be a differentiable function whose gradient vector ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

In other words: the line integral of a conservative vector field is independent of the path.

Surface Integrals

If S is given by a parametrization $\mathbf{r}(u,v)$ and f is a scalar function defined on S, then

$$\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))||\mathbf{r}_u \times \mathbf{r}_v||dudv$$

where D is the domain of the parameters u, v.

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S given by the parametrization $\mathbf{r}(u,v)$, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)dudv$$

where D is the domain of the parameters u, v.

Green's Theorem

Theorem (Green's Theorem)**.** *Let C be a positively-oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then*

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Note: We can write the left integral as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle P, Q \rangle$.

Stokes' Theorem

Stokes' Theorem is the 3D version of Green's Theorem.

Theorem (Stokes' Theorem)**.** *Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

Go left to right whenever you have a flat surface. Go right to left whenever the boundary curve is easy to parametrize (e.g. a circle).

Divergence Theorem

Theorem (Divergence Theorem)**.** *Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an opearn region that contains E. Then*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div}(\mathbf{F})dV.$$