Math 33B Notes

Written by Victoria Kala
vtkala@math.ucla.edu
Last updated April 16, 2019

Classification of Differential Equations

The **order** of a differential equation is the order of the highest derivative that appears in the equation. For example, the differential equation

$$\frac{d^3y}{dx^3} + x^2y\frac{d^5y}{dx^5} - \sin^3(xy)\frac{dy}{dx} = \cot(x)$$

has order 5.

An ordinary differential equation is said to be **linear** if it can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = g(x)$$

where $g(x)$ and each $a_i(x)$ are functions of x and not y. For example, the differential equation

$$y''' + xy^2y' = x^4$$

is not linear because of the term xy^2y'.

Separable Equations

A separable equation is a first order linear differential equation that can be written as

$$\frac{dy}{dx} = f(x)g(y).$$

Use the following steps to solve the equation:

1. Write a function of x on one side of the equation and a function of y on the other side of the equation, e.g. the above equation would look like $\frac{dy}{g(y)} = f(x)dx$.
2. Integrate both sides.
3. Solve for y if possible and C if given an initial value.

First Order Linear Equations

A first order linear differential equation is of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Use the following steps to solve the equation:

1. Write the equation in **standard form**: $\dfrac{dy}{dx} + P(x)y = f(x)$.
2. Find the integrating factor $\mu(x) = e^{\int P(x)dx}$.
3. Multiply the standard equation in Step 1 by $\mu(x)$. Your equation can now be written in the form $(\mu(x)y)' = \mu(x)f(x)$.
4. Integrate: $\mu(x)y = \int \mu(x)f(x)dx$.
5. Solve for y and C if given an initial value: $y = \dfrac{\int \mu(x)f(x)dx}{\mu(x)}$.

Friendly note: While you *could* just memorize the formula in Step 5, you shouldn't. It is better to follow the steps listed above and understand what you are solving rather than just plug in your functions into a formula.

Exact Equations and Integrating Factors

An equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \qquad \text{or} \qquad M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is said to be an **exact equation** if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Our goal is to find a function $f(x,y)$ such that:

$$\frac{\partial f}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N(x,y).$$

Use the following steps to solve an **exact** equation:

1. Verify that $M_y = N_x$.
2. Since we want $\frac{\partial f}{\partial x} = M(x,y)$, integrate with respect to x:

$$f(x,y) = \int M(x,y)dx + g(y).$$

3. Since we want $\frac{\partial f}{\partial y} = N(x,y)$, differentiate what you found in Step (2) with respect to y:

$$\frac{\partial}{\partial y}\int M(x,y)dx + g'(y) = N(x,y).$$

4. Solve for $g'(y)$:

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx$$

and integrate with respect to y on both sides.

5. Plug your solution for $g(y)$ into your equation for step (2). The solution is $f(x,y) = C$.

If $M(x,y)dx + N(x,y)dy = 0$ is not exact, it is sometimes possible to find an integrating factor $\mu(x,y)$ such that

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact. Notice that this is just the integrating factor $\mu(x,y)$ multiplied by the original equation.

Use the following to find the integrating factor μ: beginenumerate

Verify that $M_y \neq N_x$.

Find the integrating factor $\mu(x,y)$. This is done as follows:

- If $\dfrac{M_y - N_x}{N}$ is a function of x alone, then $\mu(x,y) = \exp\left(\int \dfrac{M_y - N_x}{N}dx\right)$.
- If $\dfrac{N_x - M_y}{M}$ is a function of y alone, then $\mu(x,y) = \exp\left(\int \dfrac{N_x - M_y}{M}dy\right)$.

Multiply $\mu(x)$ by your differential equation: $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$. Double check that this equation is exact by showing that $\frac{\partial(\mu M)}{dy} = \frac{\partial(\mu N)}{dx}$.

Use the steps as outlined above for solving an exact equation.

The Existence and Uniqueness Theorem (for first order equations)

Theorem (Existence and Uniqueness Theorem for First Order Linear Equations)**.** *If the functions P and f are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation $y' + P(t)y = f(t)$ for t in I, and that also satisfies the initial condition $y(t_0) = y_0$, where y_0 is an arbitrary prescribed value.*

What this means if that P and f are continuous on its domain, and your initial value t_0 is in some interval in that domain, then there exists a unique solution to that differential equation in that interval.

For example, consider the differential equation

$$y' + \frac{1}{x-1}y = \frac{1}{\sqrt{x+4}}.$$

$P(x) = \frac{1}{x-1}$ and $f(x) = \frac{1}{\sqrt{x+4}}$. The domain of $P(x)$ is everything except $x = 1$, which we can write as $(-\infty, 1) \cup (1, \infty)$. The domain of $f(x)$ is all numbers greater than -4, or $(-4, \infty)$. Taking the intersection of these domains gives us $(-4, 1) \cup (1, \infty)$. P and f are continuous on this domain. If we were given the initial value of $y(5) = y_0$, the largest interval that guarantees a solution is the interval $(1, \infty)$ since 5 is in that interval. If we were given the initial value of $y(-\pi) = y_0$, the largest interval that guarantees a solution is the interval $(-4, 1)$ since $-\pi$ is in that interval.