Final Practice Problems

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Name: ______________________________ Score: NA

Directions: You may use one 3" × 5" notecard, however no other outside resources such as books, notes, or calculators are allowed. Write your solutions in your bluebook and clearly mark the problems. If you solve a problem multiple times, cross out the work you do not want graded, otherwise you will receive little or no partial credit. Unless otherwise specified, numbers included in a solution are not to be approximated, but instead expressed as exact numbers (i.e., in terms of square roots, multiples of π, etc.).

Disclaimer: The content and level of difficulty of this quiz are not guaranteed to be in correlation with the midterm nor final examinations in any form.

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1. Find the vector $\mathbf{x}$ determined by the given coordinate vector $\mathbf{x}_B$ and given basis B:

   (a) $B = \left\{ \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ 6 \end{pmatrix} \right\}, \mathbf{x}_B = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

   (b) $B = \left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix} \right\}, \mathbf{x}_B = \begin{pmatrix} -4 \\ 8 \\ -7 \end{pmatrix}$

2. Find the coordinate vector $\mathbf{x}_B$ of the given vector $\mathbf{x}$ relative to the given basis B:

   (a) $B = \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \end{pmatrix} \right\}, \mathbf{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

   (b) $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}, \mathbf{x} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

3. Find the change of coordinates matrix from B to C and the change of coordinates matrix from C to B where
$$B = \left\{ \begin{pmatrix} -1 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \end{pmatrix} \right\}, C = \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

4. Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.

5. Find the dimension of the subspace spanned by the given vectors:
$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -8 \\ 6 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 7 \end{pmatrix}$$

6. Find the rank and nullity of $A = \begin{pmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

7. If the null space of a 5×6 matrix A is 4-dimensional, what are the dimensions of the column space and row space of A, and the dimension of the null space of A^T?

8. Show that if A has eigenvalue $\lambda = 0$ then A^{-1} does not exist.

9. Is $\lambda = 2$ an eigenvalue of $\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$? Why or why not?

10. Is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix}$? If so, find the eigenvalue.

11. Find the eigenvalues and eigenvectors of each matrix:

    (a) $\begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix}$

    (b) $\begin{pmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{pmatrix}$

12. Let A and B be similar matrices. Show that A and B have the same eigenvalues.

13. Diagonalize the matrix $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$.

14. Let $\mathbf{u} = (2, -1, 1)$ and $\mathbf{v} = (1, 1, 2)$. Find $\mathbf{u} \cdot \mathbf{v}$ and determine the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

15. Show that $\mathbf{u} = (6, 1, 4)$ and $\mathbf{v} = (2, 0, -3)$ are orthogonal.

16. Find the distance between $\mathbf{u} = (10, -3)$ and $\mathbf{v} = (-1, -5)$.

Problems added 11/30:

17. Let A be a matrix with eigenvalue $\lambda = 2$. Find the eigenvalues of the following:

    (a) A^{10}

    (b) A^{-1}

    (c) $A + 4I$

    (d) $100A$

18. Let $\mathbf{u}_1 = (0, 1, 0), \mathbf{u}_2 = (1, 0, 1), \mathbf{u}_3 = (1, 0, -1)$ be vectors in $\mathbf{R}^3$.

    (a) Show that the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set.

    (b) Convert S into an orthonormal set by normalizing the vectors.

19. In each part, an orthonormal basis $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ is given. Find the coordinate vector of $\mathbf{w}$ with respect to that basis. (Use inner products!)

   (a) $\mathbf{u}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right); \mathbf{w} = (3, 7)$

   (b) $\mathbf{u}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \mathbf{u}_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \mathbf{u}_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right); \mathbf{w} = (-1, 0, 2)$

20. Determine which of the following matrices are orthogonal:

   (a) $\begin{pmatrix} 0 & 1 & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}$

   (b) $\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$

21. Find the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$:

   (a) $\mathbf{v} = (6, 7), \mathbf{w} = (3, 4)$

   (b) $\mathbf{v} = (1, 2, 0), \mathbf{w} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

22. Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$, and find the orthogonal projection of $\mathbf{b}$ onto the column space of A:

   (a) $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 7 \\ 0 \\ -7 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ 9 \\ 3 \end{pmatrix}$