Solutions to Final Practice Problems

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Answers

This page contains answers only. See the following pages for detailed solutions.

1. (a) $\mathbf{x} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$
   (b) $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$
2. (a) $\mathbf{x}_B = \begin{pmatrix} 8 \\ -5 \end{pmatrix}$
   (b) $\mathbf{x}_B = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$
3. $P_{B\to C} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}, P_{C\to B} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
4. 2
5. 3
6. $rank(A) = 3$, $nullity(A) = 2$
7. $\dim col(A) = 2$
   $\dim row(A) = 2$
   $\dim null(A^T) = 1$
8. See detailed solution
9. Yes
10. Yes, $\lambda = -2$
11. (a) $\lambda_1 = 1 + \sqrt{2}, \lambda_2 = 1 - \sqrt{2}$
    $\mathbf{v}_1 = \begin{pmatrix} 2+\sqrt{2} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2-\sqrt{2} \\ 1 \end{pmatrix}$
    (b) $\lambda_1 = 3, \lambda_2 = 10, \lambda_3 = 5$, $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -7/11 \\ 14/11 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2/9 \\ 1/9 \\ 1 \end{pmatrix}$
12. See detailed solution
13. $A = PDP^{-1}$ where $P = \begin{pmatrix} 1 & -3/4 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$
14. $3, \dfrac{\pi}{3}$
15. See detailed solution
16. $5\sqrt{5}$
17. (a) 2^{10}
    (b) $\frac{1}{2}$
    (c) 6
    (d) 200
18. (a) See detailed solution
    (b) See detailed solution
19. (a) $\mathbf{w}_B = (-2\sqrt{2}, 5\sqrt{2})$
    (b) $\mathbf{w}_B = (0, -2, 1)$
20. (a) Not orthogonal
    (b) Orthogonal
21. (a) $\frac{46}{25}(3,4)$
    (b) $(1,1,1)$
22. (a) $\hat{\mathbf{x}} = \begin{pmatrix} 5 \\ 1/2 \end{pmatrix}$, $A\hat{\mathbf{x}} = \begin{pmatrix} 11/2 \\ -9/2 \\ -4 \end{pmatrix}$
    (b) $\hat{\mathbf{x}} = \begin{pmatrix} 12 \\ -3 \\ 9 \end{pmatrix}, A\hat{\mathbf{x}} = \begin{pmatrix} 3 \\ 3 \\ 9 \\ 0 \end{pmatrix}$

Detailed Solutions

1. Find the vector $\mathbf{x}$ determined by the given coordinate vector $\mathbf{x}_B$ and given basis B:

   (a) $B = \left\{ \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ 6 \end{pmatrix} \right\}, \mathbf{x}_B = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

   Solution. Since B is a basis of $\mathbb{R}^2$, for every $\mathbf{x} \in \mathbb{R}^2$ we can write

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$$

   where $\mathbf{b}_1, \mathbf{b}_2$ are the basis elements of B. The coordinate vector is the vector $\mathbf{x}_B = (c_1, c_2)$. Since we are given $\mathbf{x}_B = (5, 3)$, then

$$\mathbf{x} = 5\begin{pmatrix} 3 \\ -5 \end{pmatrix} + 3\begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 15 \\ -25 \end{pmatrix} + \begin{pmatrix} -12 \\ 18 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}.$$

   □

   (b) $B = \left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix} \right\}, \mathbf{x}_B = \begin{pmatrix} -4 \\ 8 \\ -7 \end{pmatrix}$

   Solution. Since B is a basis of $\mathbb{R}^3$, for every $\mathbf{x} \in \mathbb{R}^3$ we can write

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$$

   where $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are the basis elements of B. The coordinate vector is the vector $\mathbf{x}_B = (c_1, c_2, c_3)$. Since we are given $\mathbf{x}_B = (-4, 8, -7)$, then

$$\mathbf{x} = (-4)\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 8\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + (-7)\begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 24 \\ -40 \\ 16 \end{pmatrix} + \begin{pmatrix} -28 \\ 49 \\ -21 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$$

   □

2. Find the coordinate vector $\mathbf{x}_B$ of the given vector $\mathbf{x}$ relative to the given basis B:

   (a) $B = \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \end{pmatrix} \right\}, \mathbf{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

   Solution. Since B is a basis of $\mathbb{R}^2$, for every $\mathbf{x} \in \mathbb{R}^2$ we can write

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$$

   where $\mathbf{b}_1, \mathbf{b}_2$ are the basis elements of B. The coordinate vector is the vector $\mathbf{x}_B = (c_1, c_2)$. Since we are given $\mathbf{x} = (-2, 1)$, then

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -5 \end{pmatrix}.$$

We can rewrite this system as a matrix equation and use row reduction to solve (you could also use an inverse to solve):

$$\left(\begin{array}{cc|c} 1 & 2 & -2 \\ -3 & -5 & 1 \end{array}\right) \xrightarrow{3R1+R2\to R2} \left(\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -5 \end{array}\right) \xrightarrow{-2R2+R1\to R1} \left(\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -5 \end{array}\right)$$

The solution to this system is $c_1 = 8, c_2 = -5$, therefore $\mathbf{x}_B = \begin{pmatrix} 8 \\ -5 \end{pmatrix}$. □

(b) $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}, \mathbf{x} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

Solution. Since B is a basis of $\mathbb{R}^3$, for every $\mathbf{x} \in \mathbb{R}^3$ we can write

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$$

where $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are the basis elements of B. The coordinate vector is the vector $\mathbf{x}_B = (c_1, c_2, c_3)$. Since we are given $\mathbf{x} = (3, -5, 4)$, then

$$\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

We can rewrite this system as a matrix equation and use row reduction to solve (you could also use an inverse to solve, if it exists):

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{array}\right) \xrightarrow{-3R1+R3\to R3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 2 & -1 & -5 \end{array}\right) \xrightarrow{-2R2+R3\to R3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$$\xrightarrow{R3+R2\to R2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right) \xrightarrow{-R3+R1\to R1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$$\xrightarrow{-2R2+R1\to R1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

The solution to this system is $c_1 = -2, c_2 = 0, c_3 = 5$, therefore $\mathbf{x}_B = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$. □

3. Find the change of coordinates matrix from B to C and the change of coordinates matrix from C to B where

$$B = \left\{ \begin{pmatrix} -1 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \end{pmatrix} \right\}, C = \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Solution. To find the change of coordinates matrix from B to C, call it $P_{B\to C}$, we set up the augmented matrix

$$\left(\begin{array}{cc|cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array} \right),$$

where $\mathbf{b}_1, \mathbf{b}_2$ are the basis elements of B and $\mathbf{c}_1, \mathbf{c}_2$ are the basis elements of C. We then reduce to get the $\left(\begin{array}{c|c} I & P_{B\to C} \end{array} \right)$.

$$\left(\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 4 & 1 & 8 & -5 \end{array} \right) \xrightarrow{-4R1+R2\to R2} \left(\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & -3 & 12 & -9 \end{array} \right) \xrightarrow{-\frac{1}{3}R2\to R2} \left(\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & 3 \end{array} \right)$$

$$\xrightarrow{-R2+R1\to R1} \left(\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{array} \right).$$

Thus $P_{B\to C} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$.

To find the change of coordinates matrix from C to B, call it $P_{B\to C}$, we can set up the augmented matrix

$$\left(\begin{array}{cc|cc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{c}_1 & \mathbf{c}_2 \end{array} \right),$$

and solve using the same method as above, OR we can calculate $P_{B\to C}^{-1}$. I will do the latter, but feel free to use whichever method you are more comfortable with.

$$P_{C\to B} = P_{B\to C}^{-1} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}^{-1} = \frac{1}{9-8} \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}.$$

□

4. Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.

Solution. There are two ways to approach this problem: one way is to find the basis of this set, another is to look at what the elements look like. Both approaches are very similar.

We first want to write this set out:

$$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = z \right\}$$

The $x = z$ represents that the first and third entries are equal. Since $x = z$, substitute this in to either the first or third equation. Every element in U will look like

$$\begin{pmatrix} x \\ y \\ x \end{pmatrix}.$$

There are two unknowns here (x and y), hence the dimension is 2.

If you want to see what the basis looks like, write it out the element in parametric form:

$$\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} y.$$

Therefore the basis of U is the set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. This set has 2 elements, hence the dimension is 2. □

5. Find the dimension of the subspace spanned by the given vectors:

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -8 \\ 6 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 7 \end{pmatrix}$$

Solution. Let U denote the span of the set of vectors above. Recall that the dimension relates to the basis, and that a basis is the set of linearly independent vectors which span the set. To find the linearly independent vectors of U, we will form a matrix, reduce, and identify the columns with pivots (note that these are the same steps to finding the column space!).

$$\begin{pmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{pmatrix} \xrightarrow{2R1+R2\to R2} \begin{pmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 1 & 5 & 7 \end{pmatrix} \xrightarrow{R1\leftrightarrow R2} \begin{pmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 7 \\ 0 & -2 & -10 & -6 \end{pmatrix}$$

$$\xrightarrow{2R2+R3\to R3} \begin{pmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

The number of pivots will tell us the number of elements in our basis for U. Here we have 3 pivots (1st, 2nd, 4th columns), hence $\dim(U) = 3$.

If you want to write out the basis, you will just take the 1st, 2nd, and 4th vectors from U:

$$U = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 7 \end{pmatrix} \right\}$$

Notice that we have 3 elements in our basis, hence $\dim(U) = 3$. □

6. Find the rank and nullity of $A = \begin{pmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Solution. The rank is the dimension of the column space, which is just the number of pivots. We have 3 pivots (1st, 2nd, 5th columns), hence $rank(A) = 3$. Using the Rank-Nullity theorem,

$$nullity(A) = \#columns(A) - rank(A) = 5 - 3 = 2.$$

□

7. If the null space of a 5×6 matrix A is 4-dimensional, what are the dimensions of the column space and row space of A, and the dimension of the null space of A^T?

Solution. Since the nullspace of A is 4-dimensional, then $nullity(A) = 4$. Using the Rank-Nullity Theorem,

$$rank(A) = \#columns(A) - nullity(A) = 6 - 4 = 2.$$

The rank is equal to the dimension of the column space and dimension of the row space, hence these are both 2. The dimension of the null space of A^T is

$$nullity(A^T) = \#rows(A) - rank(A) = 5 - 4 = 1.$$

To summarize: $\dim col(A) = 2, \dim row(A) = 2, \dim null(A^T) = 1.$ □

8. Show that if A has eigenvalue $\lambda = 0$ then A^{-1} does not exist.

Proof. We will use the eigenvalue equation $\det(A - \lambda I) = 0$. If $\lambda = 0$, then this equation becomes

$$\det(A - 0I) = 0 \quad \Rightarrow \quad \det(A) = 0.$$

Since $\det(A) = 0$, then A is not invertible, hence A^{-1} does not exist. □

9. Is $\lambda = 2$ an eigenvalue of $\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$? Why or why not?

Solution. We will use the eigenvalue equation $\det(A - \lambda I) = 0$. We need to show that if $\lambda = 2$, then

$$\det(A - 2I) = 0.$$

Start with the left hand side and show we get 0:

$$\det(A - 2I) = \det\left(\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 1 \cdot 6 - 2 \cdot 3 = 0.$$

Since we got 0, this shows that the eigenvalue equation is satisfied. $\lambda = 2$ is indeed an eigenvalue of $\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$. □

10. Is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix}$? If so, find the eigenvalue.

Solution. We will use the eigenvalue-eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$. We need to show that if $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, then this equation is satisfied. Start with the left hand side and show we get a multiple of $\mathbf{v}$:

$$\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 - 12 + 7 \\ 3 - 6 + 7 \\ 5 - 12 + 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Since we got a multiple of $\mathbf{v}$, then $\mathbf{v}$ is an indeed an eigenvector of $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix}$, and its eigenvalue is $\lambda = -2$. □

11. Find the eigenvalues and eigenvectors of each matrix:

(a) $\begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix}$

Solution. We will use the eigenvalue equation $\det(A - \lambda I) = 0$:

$$\det\left(\begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - (-2)(1)$$
$$= -3 - 3\lambda + \lambda + \lambda^2 + 2 = \lambda^2 - 2\lambda - 1.$$

We use the quadratic formula to find the roots of this equation:

$$\lambda = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}.$$

Therefore the eigenvalues are $\lambda_1 = 1 + \sqrt{2}, \lambda_2 = 1 - \sqrt{2}$.
We now find the eigenvectors using the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$.
When $\lambda_1 = 1 + \sqrt{2}$,

$$\left(\begin{array}{cc|c} 3-\lambda_1 & -2 & 0 \\ 1 & -1-\lambda_1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 3-(1+\sqrt{2}) & -2 & 0 \\ 1 & -1-(1+\sqrt{2}) & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 2-\sqrt{2} & -2 & 0 \\ 1 & -2-\sqrt{2} & 0 \end{array}\right) \xrightarrow{-(2-\sqrt{2})R2+R1\to R1} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -2-\sqrt{2} & 0 \end{array}\right)$$

x_1 is a pivot, x_2 is a free variable. The second equation tells us that $x_1 = (2+\sqrt{2})x_2$, hence the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (2+\sqrt{2})x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{2} \\ 1 \end{pmatrix} x_2.$$

This shows that λ_1 has the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 2+\sqrt{2} \\ 1 \end{pmatrix}$.

When $\lambda_2 = 1 - \sqrt{2}$,

$$\left(\begin{array}{cc|c} 3-\lambda_2 & -2 & 0 \\ 1 & -1-\lambda_2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 3-(1-\sqrt{2}) & -2 & 0 \\ 1 & -1-(1-\sqrt{2}) & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 2+\sqrt{2} & -2 & 0 \\ 1 & -2+\sqrt{2} & 0 \end{array}\right) \xrightarrow{-(2+\sqrt{2})R2+R1\to R1} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -2+\sqrt{2} & 0 \end{array}\right)$$

x_1 is a pivot, x_2 is a free variable. The second equation tells us that $x_1 = (2-\sqrt{2})x_2$, hence the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (2-\sqrt{2})x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2-\sqrt{2} \\ 1 \end{pmatrix} x_2.$$

This shows that λ_2 has the eigenvector $\mathbf{v}_2 = \begin{pmatrix} 2-\sqrt{2} \\ 1 \end{pmatrix}$.

□

(b) $\begin{pmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{pmatrix}$

Solution. We will use the eigenvalue equation $\det(A - \lambda I) = 0$:

$$\begin{aligned}
\det\left(\begin{pmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right) &= \begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 9-\lambda & 0 \\ 5 & 8 & 3-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = (3-\lambda)\left[(6-\lambda)(9-\lambda) - (-2)(-2)\right] \\
&= (3-\lambda)\left[54 - 6\lambda - 9\lambda + \lambda^2 - 4\right] = (3-\lambda)(\lambda^2 - 15\lambda - 50) \\
&= (3-\lambda)(\lambda-10)(\lambda-5)
\end{aligned}$$

Set this equal to 0 and solve for λ. The eigenvalues are $\lambda_1 = 3, \lambda_2 = 10, \lambda_3 = 5$.
We now find the eigenvectors using the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$.
When $\lambda_1 = 3$,

$$\left(\begin{array}{ccc|c} 6-\lambda_1 & -2 & 0 & 0 \\ -2 & 9-\lambda_1 & 0 & 0 \\ 5 & 8 & 3-\lambda_1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 6-3 & -2 & 0 & 0 \\ -2 & 9-3 & 0 & 0 \\ 5 & 8 & 3-3 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ -2 & 6 & 0 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right) \xrightarrow{R1+R2\rightarrow R1} \left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ -2 & 6 & 0 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right) \xrightarrow{2R1+R2\rightarrow R2} \left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 14 & 0 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-5R1+R3\rightarrow R3} \left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 14 & 0 & 0 \\ 0 & -12 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{14}R2\rightarrow R2, -\frac{1}{12}R3\rightarrow R3} \left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-R2+R3\rightarrow R3} \left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-4R2+R1\rightarrow R1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_1, x_2 are pivots. x_3 is a free variable. The first equation tells us that $x_1 = 0$, the second equation tells us that $x_2 = 0$, hence the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_3.$$

This shows that λ_1 has the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

When $\lambda_2 = 10$,

$$\left(\begin{array}{ccc|c} 6-\lambda_2 & -2 & 0 & 0 \\ -2 & 9-\lambda_2 & 0 & 0 \\ 5 & 8 & 3-\lambda_2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 6-10 & -2 & 0 & 0 \\ -2 & 9-10 & 0 & 0 \\ 5 & 8 & 3-10 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} -4 & -2 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 5 & 8 & -7 & 0 \end{array}\right) \xrightarrow{-2R2+R1\to R1} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 5 & 8 & -7 & 0 \end{array}\right) \xrightarrow{R1\leftrightarrow R2} \left(\begin{array}{ccc|c} 5 & 8 & -7 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{2R2+R1\to R1} \left(\begin{array}{ccc|c} 1 & 6 & -7 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{2R1+R2\to R2} \left(\begin{array}{ccc|c} 1 & 6 & -7 & 0 \\ 0 & 11 & -14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{11}R2\to R2} \left(\begin{array}{ccc|c} 1 & 6 & -7 & 0 \\ 0 & 1 & -14/11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-6R2+R1\to R1} \left(\begin{array}{ccc|c} 1 & 0 & 7/11 & 0 \\ 0 & 1 & -14/11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_1, x_2 are pivots. x_3 is a free variable. The first equation tells us that $x_1 = -\frac{7}{11}x_3$, the second equation tells us that $x_2 = \frac{14}{11}x_3$, hence the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7/11x_3 \\ 14/11x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7/11 \\ 14/11 \\ 1 \end{pmatrix} x_3.$$

This shows that λ_2 has the eigenvector $\mathbf{v}_2 = \begin{pmatrix} -7/11 \\ 14/11 \\ 1 \end{pmatrix}$.

When $\lambda_3 = 5$,

$$\left(\begin{array}{ccc|c} 6-\lambda_3 & -2 & 0 & 0 \\ -2 & 9-\lambda_3 & 0 & 0 \\ 5 & 8 & 3-\lambda_3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 6-5 & -2 & 0 & 0 \\ -2 & 9-5 & 0 & 0 \\ 5 & 8 & 3-5 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 5 & 8 & -2 & 0 \end{array}\right) \xrightarrow{2R1+R2\to R2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 8 & -2 & 0 \end{array}\right) \xrightarrow{R1\leftrightarrow R3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 5 & 8 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-5R1+R2\to R2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 18 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{18}R2\to R2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & -1/9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{2R2+R1\to R1} \left(\begin{array}{ccc|c} 1 & 0 & -2/9 & 0 \\ 0 & 1 & -1/9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_1, x_2 are pivots. x_3 is a free variable. The first equation tells us that $x_1 = \frac{2}{9}x_3$, the second equation tells us that $x_2 = \frac{1}{9}x_3$, hence the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2/9x_3 \\ 1/9x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2/9 \\ 1/9 \\ 1 \end{pmatrix} x_3.$$

This shows that λ_3 has the eigenvector $\mathbf{v}_3 = \begin{pmatrix} 2/9 \\ 1/9 \\ 1 \end{pmatrix}$.

□

12. Let A and B be similar matrices. Show that A and B have the same eigenvalues.

Proof. Since A and B are similar, there exists an invertible P such that $A = PBP^{-1}$. To show that the eigenvalues of A and B are the same, we wish to show that $\det(A - \lambda I) = \text{B} - \lambda\text{I}$. Starting from the left hand side,

$$\begin{aligned}\det(A - \lambda I) &= \det(PBP^{-1} - \lambda I) = \det(PBP^{-1} - \lambda PP^{-1}) = \det\left[P(BP^{-1} - \lambda P^{-1})\right] \\ &\det\left[P(B - \lambda I)P^{-1}\right] = \det(P)\det(B - \lambda I)\det(P^{-1}) = \det(P)\det(B - \lambda I)\frac{1}{\det(P)} \\ &= \det(B - \lambda I)\end{aligned}$$

Since we have shown that $\det(A - \lambda I) = \det(B - \lambda I)$, then A and B must have the same eigenvalues. □

13. Diagonalize the matrix $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$.

Solution. We begin by finding the eigenvalues. We will use the eigenvalue equation $\det(A - \lambda I) = 0$:

$$\begin{aligned}\det\left(\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) &= \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 12 \\ &= 2 - 2\lambda - \lambda + \lambda^2 - 12 = \lambda^2 - 3\lambda - 10 \\ &= (\lambda - 5)(\lambda + 2)\end{aligned}$$

Set this equal to 0 and solve for λ. The eigenvalues are $\lambda_1 = 5$, $\lambda_2 = -2$.

We now find the eigenvectors using the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$:

When $\lambda_1 = 5$:

$$\left(\begin{array}{cc|c} 2-\lambda_1 & 3 & 0 \\ 4 & 1-\lambda_1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2-5 & 3 & 0 \\ 4 & 1-5 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R1 \to R1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 4 & -4 & 0 \end{array}\right) \xrightarrow{-4R1+R2 \to R2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

x_1 is a pivot, x_2 is a free variable. The first equation tells us that $x_1 = x_2$, hence the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2.$$

This shows that λ_1 has eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

When $\lambda_2 = -2$:

$$\left(\begin{array}{cc|c} 2-\lambda_2 & 3 & 0 \\ 4 & 1-\lambda_2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2-(-2) & 3 & 0 \\ 4 & 1-(-2) & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 4 & 3 & 0 \\ 4 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-R1+R2\to R2} \left(\begin{array}{cc|c} 4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{4}R1\to R1} \left(\begin{array}{cc|c} 1 & 3/4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

x_1 is a pivot, x_2 is a free variable. The first equation tells us that $x_1 = -3/4x_2$, hence the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3/4x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3/4 \\ 1 \end{pmatrix} x_2.$$

This shows that λ_2 has the eigenvector $\mathbf{v}_2 = \begin{pmatrix} -3/4 \\ 1 \end{pmatrix}$.

We now form the matrix P of eigenvectors and the diagonal matrix D of eigenvalues. You get to choose the order. I will choose $P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$, which means that λ_1 and λ_2 will follow the same order in D:

$$P = \begin{pmatrix} 1 & -3/4 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}.$$

We now calculate P^{-1}. You can either set up an augmented matrix or use the equation of the inverse of a 2×2 matrix. I will do the latter:

$$P^{-1} = \frac{1}{(1)(1) - (-3/4)(1)} \begin{pmatrix} 1 & 3/4 \\ -1 & 1 \end{pmatrix} = \frac{4}{7} \begin{pmatrix} 1 & 3/4 \\ -1 & 1 \end{pmatrix}.$$

If our calculations are correct, $A = PDP^{-1}$. Let's check:

$$\begin{aligned} PDP^{-1} &= \begin{pmatrix} 1 & -3/4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \frac{4}{7} \begin{pmatrix} 1 & 3/4 \\ -1 & 1 \end{pmatrix} \\ &= \frac{4}{7} \begin{pmatrix} 1 & -3/4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3/4 \\ -1 & 1 \end{pmatrix} \\ &= \frac{4}{7} \begin{pmatrix} 5 & 6/4 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3/4 \\ -1 & 1 \end{pmatrix} \\ &= \frac{4}{7} \begin{pmatrix} 5 - 6/4 & 15/4 + 6/4 \\ 5 + 2 & 15/4 - 2 \end{pmatrix} = \frac{4}{7} \begin{pmatrix} 14/4 & 21/7 \\ 7 & 7/4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \\ &= A \end{aligned}$$

□

14. Let $\mathbf{u} = (2, -1, 1)$ and $\mathbf{v} = (1, 1, 2)$. Find $\mathbf{u} \cdot \mathbf{v}$ and determine the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

Solution. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = 2(1) - 1(1) + 1(2) = 2 - 1 + 2 = 3.$$

To determine θ, we will use the fact that

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta \quad \Rightarrow \quad \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}.$$

We now need to find $||\mathbf{u}||, ||\mathbf{v}||$:

$$\begin{aligned} ||\mathbf{u}|| &= \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6} \\ ||\mathbf{v}|| &= \sqrt{1^1 + 1^2 + 2^2} = \sqrt{6} \end{aligned}$$

Therefore

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

□

15. Show that $\mathbf{u} = (6, 1, 4)$ and $\mathbf{v} = (2, 0, -3)$ are orthogonal.

Solution. To show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, we need to show that $\mathbf{u} \cdot \mathbf{v} = 0$:

$$\mathbf{u} \cdot \mathbf{v} = 6(2) + 1(0) + 4(-3) = 12 + 0 - 12 = 0.$$

□

16. Find the distance between $\mathbf{u} = (10, -3)$ and $\mathbf{v} = (-1, -5)$.

Solution. The distance between $\mathbf{u}$ and $\mathbf{v}$ is given by $||\mathbf{u} - \mathbf{v}||$:

$$\begin{aligned} ||\mathbf{u} - \mathbf{v}|| &= ||(10, -3) - (-1, -5)|| = ||(10 - (-1), -3 - (-5))|| = ||(11, 2)|| \\ &= \sqrt{11^2 + 2^2} = \sqrt{121 + 4} = \sqrt{125} = 5\sqrt{5}. \end{aligned}$$

□

Problems added 11/30:

17. Let A be a matrix with eigenvalue $\lambda = 2$. Find the eigenvalues of the following:

 (a) A^{10}

Solution. We will use the eigenvalue-eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$. Since A has eigenvalue $\lambda = 2$, this equation becomes

$$A\mathbf{v} = 2\mathbf{v}.$$

There are two ways to do this problem: You can " build up" from the above equation, or you can start with A^{10} and "break it down".
"Build up": Multiply by A on the left on both sides:

$$A(A\mathbf{v} = 2\mathbf{v}) \quad \Rightarrow \quad A^2\mathbf{v} = 2(A\mathbf{v}) \quad \Rightarrow \quad A^2\mathbf{v} = 2(2\mathbf{v}) \quad \Rightarrow \quad A^2\mathbf{v} = 2^2\mathbf{v}.$$

Multiply by A on the left on both sides again:

$$A(A^2\mathbf{v} = 2^2\mathbf{v}) \quad \Rightarrow \quad A^3\mathbf{v} = 2^2(A\mathbf{v}) \quad \Rightarrow A^3\mathbf{v} = 2^2(2\mathbf{v}) \quad \Rightarrow \quad A^3\mathbf{v} = 2^3\mathbf{v}.$$

Repeat this pattern of multiplying by A on the left. You will get

$$A^{10}\mathbf{v} = 2^{10}\mathbf{v}.$$

So A^{10} has eigenvalue 2^{10}.
"Break it down": Start with $A^{10}\mathbf{v}$:

$$\begin{aligned} A^{10}\mathbf{v} &= A \cdot A \cdots A \cdot (A\mathbf{v}) = A \cdot A \cdots Acdot2\mathbf{v}) \\ &= 2A \cdot A \cdot A \cdot (A\mathbf{v}) = 2A \cdot Ac \ldots A \cdot 2\mathbf{v} \\ &= 2^2 A \cdot A \cdots A\mathbf{v} \\ &\vdots \\ &= 2^{10}\mathbf{v}. \end{aligned}$$

Again, A^{10} has eigenvalue 2^{10}. □

(b) A^{-1}

Solution. We will again use the equation $A\mathbf{v} = 2\mathbf{v}$. Multiply by A^{-1} on both sides:

$$A^{-1}(A\mathbf{v} = 2\mathbf{v}) \quad \Rightarrow \quad \mathbf{v} = 2A^{-1}\mathbf{v} \quad \Rightarrow \quad A^{-1}\mathbf{v} = \frac{1}{2}\mathbf{v}$$

Thus A^{-1} has eigenvalue $\frac{1}{2}$. □

(c) $A + 4I$

Solution. We will again use the equation $A\mathbf{v} = 2\mathbf{v}$. Add $4I\mathbf{v}$ onto both sides:

$$A^{\mathbf{v}} + 4I\mathbf{v} = 2\mathbf{v} + 4I\mathbf{v} \quad \Rightarrow \quad (A + 4I)\mathbf{v} = 2\mathbf{v} + 4\mathbf{v} \quad \Rightarrow \quad (A + 4I)\mathbf{v} = 6\mathbf{v}.$$

Thus $A + 4I$ has eigenvalue 6. □

(d) $100A$

Solution. We will again use the equation $A\mathbf{v} = 2\mathbf{v}$. Multiply by 100 on both sides:

$$100(A\mathbf{v} = 2\mathbf{v}) \quad \Rightarrow \quad 100A\mathbf{v} = 200\mathbf{v}.$$

Thus $100A$ has eigenvalue 200. □

18. Let $\mathbf{u}_1 = (0, 1, 0), \mathbf{u}_2 = (1, 0, 1), \mathbf{u}_3 = (1, 0, -1)$ be vectors in $\mathbf{R}^3$.

(a) Show that the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set.

Proof. We need to show that the dot product of all possible combinations of these vectors is 0:

$$\begin{aligned}
\mathbf{u}_1 \cdot \mathbf{u}_2 &= 0(1) + 1(0) + 0(1) = 0 \\
\mathbf{u}_1 \cdot \mathbf{u}_3 &= 0(1) + 1(0) + 0(-1) = 0 \\
\mathbf{u}_2 \cdot \mathbf{u}_3 &= 1(1) + 0(0) + 1(-1) = 1 - 1 = 0
\end{aligned}$$

Thus S is an orthogonal set. □

(b) Convert S into an orthonormal set by normalizing the vectors.

Solution. An orthonormal set is an orthogonal set and contains unit vectors. We showed in part (a) that S is an orthogonal set. Now we need to normalize the vectors.
First, look at $\mathbf{u}_1$:

$$||\mathbf{u}_1|| = \sqrt{0^2 + 1^2 + 0} = 1$$

Thus $\mathbf{u}_1$ is already a unit vector and we do not need to normalize it.
Next, look at $\mathbf{u}_2$:

$$||\mathbf{u}_2|| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$\mathbf{u}_2$ is therefore not a unit vector. We define a new vector $\mathbf{v}_2$ that is a unit vector:

$$\mathbf{v}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||} = \frac{(1,0,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$$

Now look at $\mathbf{u}_3$:

$$||\mathbf{u}_3|| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

$\mathbf{u}_3$ is therefore not a unit vector. We define a new vector $\mathbf{v}_3$ that is a unit vector:

$$\mathbf{v}_3 = \frac{\mathbf{u}_3}{||\mathbf{u}_3||} = \frac{(1,0,-1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right).$$

Thus our orthonormal set is given by $S = \{\mathbf{u}_1, \mathbf{v}_2, \mathbf{v}_3\}$. □

19. In each part, an orthonormal basis $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ is given. Find the coordinate vector of $\mathbf{w}$ with respect to that basis. (Use inner products!)

(a) $\mathbf{u}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right); \mathbf{w} = (3, 7)$

Solution. Let B denote the basis given. Since B is a basis, then for every $\mathbf{w}$ we can write

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

The coordinate vector is $\mathbf{w}_B = (c_1, c_2)$. Since B is an orthonormal basis, then we know that

$$c_1 = \frac{\mathbf{w}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1} \quad \text{and} \quad c_2 = \frac{\mathbf{w}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}.$$

These constants are called the scalar projection of $\mathbf{w}$ onto $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively. Calculate the following dot products:

$$\begin{aligned}
\mathbf{w}\cdot\mathbf{u}_1 &= 3\left(\frac{1}{\sqrt{2}}\right) + 7\left(-\frac{1}{\sqrt{2}}\right) = \frac{3-7}{\sqrt{2}} = \frac{-4}{\sqrt{2}} = -\frac{2\cdot 2}{\sqrt{2}} = -2\sqrt{2} \\
\mathbf{u}_1\cdot\mathbf{u}_1 &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1 \\
\mathbf{w}\cdot\mathbf{u}_2 &= 3\left(\frac{1}{\sqrt{2}}\right) + 7\left(\frac{1}{\sqrt{2}}\right) = \frac{3+7}{\sqrt{2}} = \frac{10}{\sqrt{2}} = \frac{5\cdot 2}{\sqrt{2}} = 5\sqrt{2} \\
\mathbf{u}_2\cdot\mathbf{u}_2 &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

Note: We should not be surprised that $\mathbf{u}_1\cdot\mathbf{u}_1$ and $\mathbf{u}_2\cdot\mathbf{u}_2$ were both 1. Recall that the definition of the norm is $||\mathbf{u}|| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$, or, equivalently, $\mathbf{u}\cdot\mathbf{u} = ||\mathbf{u}||^2$. Since we were given that B is an orthonormal basis, the norms of $\mathbf{u}_1$ and $\mathbf{u}_2$ are 1, therefore $\mathbf{u}_1\cdot\mathbf{u}_1$ and $\mathbf{u}_2\cdot\mathbf{u}_2$ are both 1.

Now we use the formulas for c_1 and c_2:

$$\begin{aligned}
c_1 &= \frac{\mathbf{w}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1} = \frac{-2\sqrt{2}}{1} = -2\sqrt{2} \\
c_2 = c_2 &= \frac{\mathbf{w}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2} = \frac{5\sqrt{2}}{1} = 5\sqrt{2}.
\end{aligned}$$

Thus $\mathbf{w}_B = (-2\sqrt{2}, 5\sqrt{2})$. □

(b) $\mathbf{u}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \mathbf{u}_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \mathbf{u}_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right); \mathbf{w} = (-1, 0, 2)$

Solution. Let B denote the basis given. Since B is a basis, then for every $\mathbf{w}$ we can write

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

The coordinate vector is $\mathbf{w}_B = (c_1, c_2, c_3)$. Since B is an orthonormal basis, then we know that

$$c_1 = \frac{\mathbf{w}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1} \quad \text{and} \quad c_2 = \frac{\mathbf{w}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2} \quad \text{and} \quad c_3 = \frac{\mathbf{w}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}.$$

These constants are called the scalar projection of $\mathbf{w}$ onto $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, respectively. Calculate the following dot products:

$$\begin{aligned}
\mathbf{w}\cdot\mathbf{u}_1 &= -1\left(\frac{2}{3}\right)+0\left(-\frac{2}{3}\right)+2\left(\frac{1}{3}\right)=0\\
\mathbf{u}_1\cdot\mathbf{u}_1 &= \frac{2}{3}\left(\frac{2}{3}\right)-\frac{2}{3}\left(-\frac{2}{3}\right)+\frac{1}{3}\left(\frac{1}{3}\right)=\frac{4}{9}+\frac{4}{9}+\frac{1}{9}=1\\
\mathbf{w}\cdot\mathbf{u}_2 &= -1\left(\frac{2}{3}\right)+0\left(\frac{1}{3}\right)+2\left(-\frac{2}{3}\right)=-\frac{2}{3}+0-\frac{4}{3}=-2\\
\mathbf{u}_2\cdot\mathbf{u}_2 &= \frac{2}{3}\left(\frac{2}{3}\right)+\frac{1}{3}\left(\frac{1}{3}\right)-\frac{2}{3}\left(-\frac{2}{3}\right)=\frac{4}{9}+\frac{1}{9}+\frac{4}{9}=1\\
\mathbf{w}\cdot\mathbf{u}_3 &= -1\left(\frac{1}{3}\right)+0\left(\frac{2}{3}\right)+2\left(\frac{2}{3}\right)=-\frac{1}{3}+0+\frac{4}{3}=1\\
\mathbf{u}_3\cdot\mathbf{u}_3 &= \frac{1}{3}\left(\frac{1}{3}\right)+\frac{2}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(\frac{2}{3}\right)=\frac{1}{9}+\frac{4}{9}+\frac{4}{9}=1
\end{aligned}$$

Again, we should not be surprised that $\mathbf{u}_1\cdot\mathbf{u}_1$, $\mathbf{u}_2\cdot\mathbf{u}_2$, and $\mathbf{u}_3\cdot\mathbf{u}_3$ were all equal to 1 (see the note in part (a)).

Now we use the formulas for c_1, c_2, c_3:

$$\begin{aligned}
c_1 &= \frac{\mathbf{w}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}=\frac{0}{1}=0\\
c_2 &= \frac{\mathbf{w}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}=\frac{-2}{1}=-2\\
c_3 &= \frac{\mathbf{w}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}=\frac{1}{1}=1
\end{aligned}$$

Thus $\mathbf{w}_B=(0,-2,1)$. □

20. Determine which of the following matrices are orthogonal:

(a) $\begin{pmatrix}0 & 1 & 1/\sqrt{2}\\ 1 & 0 & 0\\ 0 & 0 & 1/\sqrt{2}\end{pmatrix}$

Solution. Let Q denote the matrix above. We wish to show that $Q^TQ=I$.

$$\begin{aligned}
Q^TQ &= \begin{pmatrix}0 & 1 & 1/\sqrt{2}\\ 1 & 0 & 0\\ 0 & 0 & 1/\sqrt{2}\end{pmatrix}^T\begin{pmatrix}0 & 1 & 1/\sqrt{2}\\ 1 & 0 & 0\\ 0 & 0 & 1/\sqrt{2}\end{pmatrix}\\
&= \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}0 & 1 & 1/\sqrt{2}\\ 1 & 0 & 0\\ 0 & 0 & 1/\sqrt{2}\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1\end{pmatrix}
\end{aligned}$$

Since $Q^TQ \neq I$, then this matrix not orthogonal. □

(b) $\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$

Solution. Let Q denote the matrix above. We wish to show that $Q^TQ = I$.

$$\begin{aligned}
Q^TQ &= \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}^T \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \\
&= \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \\
&= \begin{pmatrix} 1/2 + 1/2 & -1/\sqrt{12} + 1/\sqrt{12} & -1/\sqrt{6} + 1/\sqrt{6} \\ -1/\sqrt{12} + 1/\sqrt{12} & 1/6 + 4/6 + 1/6 & 1/\sqrt{18} - 2/\sqrt{18} + 1/\sqrt{18} \\ -1/\sqrt{6} + 1/\sqrt{6} & 1/\sqrt{18} - 2/\sqrt{18} + 1/\sqrt{18} & 1/3 + 1/3 + 1/3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Since $Q^TQ = I$ then Q is orthogonal. □

21. Find the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$:

(a) $\mathbf{v} = (6, 7), \mathbf{w} = (3, 4)$

Solution. The orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$ is given by

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w}$$

Calculate the following dot products:

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= 6(3) + 7(4) = 18 + 28 = 46 \\
\mathbf{w} \cdot \mathbf{w} &= 3(3) + 4(4) = 9 + 16 = 25
\end{aligned}$$

Hence

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w} = \frac{46}{25}(3, 4).$$

□

(b) $\mathbf{v} = (1, 2, 0), \mathbf{w} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Solution. The orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$ is given by

$$\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$$

Calculate the following dot products:

$$\mathbf{v}\cdot\mathbf{w} = 1\left(\frac{1}{\sqrt{3}}\right) + 2\left(\frac{1}{\sqrt{3}}\right) + 0\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$$
$$\mathbf{w}\cdot\mathbf{w} = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

Hence

$$\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} = \frac{\sqrt{3}}{1}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = (1, 1, 1).$$

□

22. Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$, and find the orthogonal projection of $\mathbf{b}$ onto the column space of A:

(a) $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 7 \\ 0 \\ -7 \end{pmatrix}$

Solution. Since we are asked to find the least squares solution, we need to solve the system

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$$

for $\hat{\mathbf{x}}$. Find $A^T A$:

$$\begin{aligned} A^T A &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{pmatrix}^T \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1+1+1 & 1-1-2 \\ 1-1-2 & 1+1+4 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} \end{aligned}$$

Find $A^T\mathbf{b}$:

$$\begin{aligned} A^T\mathbf{b} &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{pmatrix}^T \begin{pmatrix} 7 \\ 0 \\ -7 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 0 \\ -7 \end{pmatrix} \\ &= \begin{pmatrix} 7+0+7 \\ 7+0-14 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ -7 \end{pmatrix} \end{aligned}$$

Therefore the system $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$ is given by

$$\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} \hat{\mathbf{x}} = \begin{pmatrix} 14 \\ -7 \end{pmatrix}$$

You can solve this using row reduction or finding an inverse matrix. Let's do row reduction:

$$\left(\begin{array}{cc|c} 3 & -2 & 14 \\ -2 & 6 & -7 \end{array} \right) \xrightarrow{R1+R2 \to R1} \left(\begin{array}{cc|c} 1 & 4 & 7 \\ -2 & 6 & -7 \end{array} \right) \xrightarrow{2R1+R2 \to R2} \left(\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & 14 & 7 \end{array} \right)$$

$$\xrightarrow{\frac{1}{14}R2 \to R2} \left(\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & 1 & 1/2 \end{array} \right) \xrightarrow{-4R2+R1 \to R1} \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 1/2 \end{array} \right)$$

therefore $\hat{\mathbf{x}} = (5, 1/2)$. This is the least squares solution.
We now wish to find the orthogonal projection of $\mathbf{b}$ onto the column space of A. We do this by finding $A\hat{\mathbf{x}}$:

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 5+1/2 \\ -5+1/2 \\ -5+1 \end{pmatrix} = \begin{pmatrix} 11/2 \\ -9/2 \\ -4 \end{pmatrix}.$$

□

(b) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ 9 \\ 3 \end{pmatrix}$

Solution. Since we are asked to find the least squares solution, we need to solve the system

$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$$

for $\hat{\mathbf{x}}$. Find A^TA:

$$\begin{aligned}
A^TA &= \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1+4+1+1 & 0+2+1+1 & -1-4+0-1 \\ 0+2+1+1 & 0+1+1+1 & 0-2+0-1 \\ -1-4+0-1 & 0-2+0-1 & 1+4+0+1 \end{pmatrix} \\
&= \begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}
\end{aligned}$$

Find $A^T\mathbf{b}$:

$$\begin{aligned}
A^T\mathbf{b} &= \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}^T \begin{pmatrix} 6 \\ 0 \\ 9 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 9 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 6+0+9+3 \\ 0+0+9+3 \\ -6+0+0-3 \end{pmatrix} \\
&= \begin{pmatrix} 18 \\ 12 \\ -9 \end{pmatrix}
\end{aligned}$$

Therefore the system $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$ is given by

$$\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix} \hat{\mathbf{x}} = \begin{pmatrix} 18 \\ 12 \\ -9 \end{pmatrix}.$$

We will solve this system using row reduction:

$$\left(\begin{array}{ccc|c} 7 & 4 & -6 & 18 \\ 4 & 3 & -3 & 12 \\ -6 & -3 & 6 & -9 \end{array}\right) \xrightarrow{R1+R3\to R1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 4 & 3 & -3 & 12 \\ -6 & -3 & 6 & -9 \end{array}\right)$$

$$\xrightarrow{-4R1+R2\to R2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 0 & -1 & -3 & -24 \\ -6 & -3 & 6 & -9 \end{array}\right) \xrightarrow{6R1+R3\to R3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 0 & -1 & -3 & -24 \\ 0 & 3 & 6 & 45 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R3\to R3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 0 & -1 & -3 & -24 \\ 0 & 1 & 2 & 15 \end{array}\right) \xrightarrow{R2+R3\to R3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 0 & -1 & -3 & -24 \\ 0 & 0 & -1 & -9 \end{array}\right)$$

$$\xrightarrow{-R3\to R3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 0 & -1 & -3 & -24 \\ 0 & 0 & 1 & 9 \end{array}\right) \xrightarrow{3R3+R2\to R2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{array}\right)$$

$$\xrightarrow{R1+R2\to R1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{array}\right) \xrightarrow{-R2\to R2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{array}\right)$$

therefore $\hat{\mathbf{x}} = (12, -3, 9)$. This is the least squares solution.

We now wish to find the orthogonal projection of $\mathbf{b}$ onto the column space of A. We do this by finding $A\hat{\mathbf{x}}$:

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 12 \\ -3 \\ 9 \end{pmatrix} = \begin{pmatrix} 12+0-9 \\ 24-3-18 \\ 12-3+0 \\ 12-3-9 \end{pmatrix} = \begin{pmatrix} 12+0-9 \\ 24-3-18 \\ 12-3+0 \\ 12-3-9 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 9 \\ 0 \end{pmatrix}.$$

□