Final Review

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Summary

This review contains notes on sections 4.4−4.7, 5.1−5.3, 6.1, 6.2, 6.5. For your final, you should...

- Know the previous material we have covered
- Know how to find the coordinates of a vector relative to a given basis
- Know how to find a vector in a given coordinate system given the vector relative to a given basis
- Know how to find perform a change of basis given two or more bases
- Know how to find the dimension of a vector space
- Know how to find the rank and nullity of a matrix
- Know how to identify if a scalar value is an eigenvalue of a matrix
- Know how to identify if a vector is a an eigenvector of a matrix
- Know how to find eigenvalues and eigenvectors
- Know how to diagonalize a matrix
- Know how to find a power of a diagonal matrix
- Know how to find the dot product of two vectors
- Know how to find the norm or length of a vector
- Know how to find the distance between two vectors
- Know how to determine if two vectors are orthogonal to each other
- Know how to find the angle between two vectors
- Know how to determine if a set is orthogonal or orthonormal
- Know how to determine if a matrix is orthogonal
- Know how to find the orthogonal projection of one vector onto another
- Know how to find a coordinate vector of a vector with respect to a basis using dot products
- Know how to find the least squares solution of a linear system

If you are not sure if you know how to do any of the above, you should read the appropriate notes and do some practice problems from your homework and textbook.

Coordinate Systems and Change of Basis

See sections 4.4, 4.7 of your textbook.

Since a basis spans a space, then all the vectors in that space can be written as a linear combination of the basis vectors, i.e. fora basis $B = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ of a vector space V, then for all $\mathbf{x} \in V$,

$$\mathbf{x} = c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n.$$

The vector of the weights $c_1, ..., c_n$ is said to be the coordinate vector of $\mathbf{x}$ with respect to B, i.e.

$$\mathbf{x}_B = \begin{pmatrix} c_1 & c_2 & ... & c_n \end{pmatrix}.$$

Dimension of a Vector Space

See sections 4.5, 4.6 of your textbook.

The **dimension** of a vector space is the number of elements in its basis. In particular, the dimension of $Null(A)$, or **nullity**, is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$ and the dimension of $Col(A)$, or **rank**, is the number of pivot columns of A.

If U is a subspace of a vector space V, then $\dim U \leq \dim V$.

We can find a way to change between bases. Let $B = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n\}$ be bases of a vector space V. Then there exists an $n \times n$ matrix $P_{B\to C}$ such that

$$\mathbf{x}_c = P_{B\to C}\mathbf{x}_B \quad \text{and} \quad \mathbf{x}_B = P_{C\to B}\mathbf{x}_C$$

where $P^{-1}_{B\to C} = P_{C\to B}$. $P_{B\to C}$ is called the **change of coordinates matrix from** B **to** C. To find $P_{B\to C}$, set up the augmented matrix

$$\left(\begin{array}{cccc|cccc} \mathbf{c}_1 & \mathbf{c}_n & ... & \mathbf{c}_n & \mathbf{b}_1 & \mathbf{b}_2 & ... & \mathbf{b}_n \end{array} \right)$$

and reduce to get

$$\left(\begin{array}{c|c} I & P_{B\to C} \end{array} \right).$$

Theorem (Rank-Nullity Theorem)**.** *If A is an $m \times n$ matrix then*

$$rank(A) + nullity(A) = n.$$

If A is an $m \times n$ matrix and has rank r, then

dim of	is...
$Col(A)$	r
$Row(A)$	r
$Null(A)$	$n - r$
$Null(A^T)$	$m - r$

Eigenvalues and Eigenvectors

See sections 5.1, 5.2 of your textbook.

λ is said to be an **eigenvalue** of a square matrix A with nonzero **eigenvector x** such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

We can rearrange this equation to get

$$(A - \lambda I)\mathbf{x} = \mathbf{0}. \tag{1}$$

Since $\mathbf{x}$ is nonzero, it must be that

$$\det(A - \lambda I) = 0. \tag{2}$$

We use equation (2) to find the eigenvalues of a matrix. After we find the eigenvalues, we then use equation (1) to find their eigenvectors. Equation (2) should yield a polynomial equation, this is sometimes called the **characteristic equation**.

A matrix A is said to be **similar** to a matrix B if there exists an invertible matrix P such that

$$A = PBP^{-1}.$$

Some nice facts:

- The eigenvalues of a triangular or diagonal matrix are the entries on its main diagonal.
- If a matrix has eigenvalue 0 then it is not invertible. (See practice problems.)
- If two matrices are similar then they have the same eigenvalues. (See practice problems.)
- The set of eigenvectors of a matrix (called the **eigenspace**) form a subspace.

Diagonalization

See section 5.3 of your textbook.

Sometimes we wish to find powers of matrices, like A^{100}. But it would be a very difficult and tedious process to multiply a matrix A out 100 times. To make this process easier we **diagonalize** a matrix: If A is an $n \times n$ matrix and has n distinct eigenvectors, then $A = PDP^{-1}$ where P is the matrix of eigenvectors of A:

$$P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & ... & \mathbf{v}_n \end{pmatrix}$$

and D is a diagonal matrix of eigenvalues of A:

$$D = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_m \end{pmatrix}.$$

If $A = PDP^{-1}$ then

$$\begin{aligned} A^k &= A \cdot A \cdots A \\ &= PDP^{-1} \cdot PDP^{-1} \cdots PDP^{-1} \\ &= PD(P^{-1}P)D(P^{-1}P) \cdots (P^{-1}P)DP^{-1} \\ &= PDD \cdots DP^{-1} \\ &= PD^kP^{-1}. \end{aligned}$$

If

$$D = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_m \end{pmatrix}$$

then

$$D^k = \begin{pmatrix} \lambda_1^k & 0 & ... & 0 \\ 0 & \lambda_2^k & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_m^k \end{pmatrix}.$$

Dot Product, Length, and Orthogonality

See section 6.1 of your textbook.

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The **dot product**, or **inner product**, $\mathbf{u} \cdot \mathbf{v}$ is defined to be

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \begin{pmatrix} u_1 & u_2 & ... & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1v_1 + u_2v_2 + ... + u_nv_n.$$

The **norm**, or **length**, of a vector $\mathbf{u} \in \mathbb{R}^n$ is defined by

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + ... + u_n^2}.$$

Notice that from this definition we have $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$.

The **distance** between two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is given by $||\mathbf{u} - \mathbf{v}||$. The vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$.

The dot product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is also given by the formula

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta$$

where θ is the angle between $\mathbf{u}, \mathbf{v}$. We can rearrange this formula to find the angle between to vectors:

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}\right).$$

Properties:

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (this says order doesn't matter)
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$
- $||c\mathbf{v}|| = |c|||\mathbf{v}||$ for any $c \in \mathbb{R}$

Orthogonal Sets

See section 6.2 of your textbook.

A set of vectors $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ is said to be an **orthogonal set** if each set of vectors is orthogonal, that is, $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for all $i \neq j$. A basis is said to be an **orthogonal basis** if it is also an orthogonal set.

Let $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y} \in W$,

$$\mathbf{y} = c_1\mathbf{u}_1 + ... + c_n\mathbf{u}_n$$

where $c_j = \dfrac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$, $j = 1, ..., p$. The term

$$\frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}\mathbf{u}_j$$

is said to be the **projection of y onto** $\mathbf{u}_j$.

A **unit vector u**, sometimes called a **normed** vector, is a vector that has length one; that is $||\mathbf{u}|| = 1$. To normalize a vector $\mathbf{v}$, we divide the vector by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}.$$

An **orthonormal** set is a set of orthogonal unit vectors. A matrix A is said to be **orthogonal** if and only if its column vectors are orthonormal; that is,

$$A^T A = I$$

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Least Squares Problems

See section 6.5 of your textbook.

The least squares solution of the system $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ such that

$$||\mathbf{b} - A\hat{\mathbf{x}}|| \leq ||\mathbf{b} - A\mathbf{x}||$$

for all $\mathbf{x}$. To find the least squares solution, we solve the system

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$$

for $\hat{\mathbf{x}}$. The orthogonal projection of $\mathbf{b}$ on the column space of A is the product $A\hat{\mathbf{x}}$.