Midterm 1 Practice Problems

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Answers

This page contains answers only. Detailed solutions are on the following pages. Please note that the detailed solutions include more work than you will need to do, I just wanted to include all the steps for your practice.

1. (a) $x_1 = 3, x_2 = 1, x_3 = 2$
   (b) $x = -\frac{3}{7}z - \frac{1}{7}, y = -\frac{4}{7}z + \frac{1}{7}, z = z$
   (c) No solution
2. (a) $\begin{pmatrix} -4 \\ 3 \\ 6 \end{pmatrix}$
   (b) Yes, $(3 - c_3)\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + (1 + c_3)\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 9 \end{pmatrix}$ where c_3 is any real number
   (c) No
   (d) $\mathbf{b}_1 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\mathbf{b}_2 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
   (e) No
3. Yes
4. (a) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} 3 \\ 4 \\ 0 \\ -9 \\ 1 \end{pmatrix} t$ where s, t are any real number
   (b) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u$ where s, t, u are any real number
5. (a) Linearly independent
   (b) Linearly independent
   (c) Linearly independent
   (d) Linearly dependent

1. Solve the following systems using row operations:

   (a) $x_1 + x_2 + 2x_3 = 8$
   $-x_1 - 2x_2 + 3x_3 = 1$
   $3x_1 - 7x_2 + 4x_3 = 10$

   Solution. Write as the augmented matrix and use row operations to solve:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right) \xrightarrow{R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{array}\right) \xrightarrow{-3R_1+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right)$$

$$\xrightarrow{-R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right) \xrightarrow{10R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{52}R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{5R_3+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{-2R_3+R_1\to R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1-2R_2\to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

   This last matrix shows that $x_1 = 3, x_2 = 1, x_3 = 2$ is the unique solution. □

   (b) $2x + 2y + 2z = 0$
   $-2x + 5y + 2z = 1$
   $8x + y + 4z = -1$

   Solution. Write as the augmented matrix and use row operations to solve:

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right) \xrightarrow{R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right) \xrightarrow{-4R_1+R_3\to R_3} \left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right)$$

$$\xrightarrow{R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1\to R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{7}R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-R_2+R_1\to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

   We have a row of zeros, and so we are missing a pivot in the third column. So z is a free variable, let $z = z$. From the second row, we have that

$$y + \frac{4}{7}z = \frac{1}{7} \Rightarrow y = -\frac{4}{7}z + \frac{1}{7}.$$

   From the first row, we have that

$$x + \frac{3}{7}z = -\frac{1}{7} \Rightarrow x = -\frac{3}{7}z - \frac{1}{7}.$$

Thus the solution is $x = -\frac{3}{7}z - \frac{1}{7}, y = -\frac{4}{7}z + \frac{1}{7}, z = z$.
Author's note: I want to point out here that I didn't get a 1 in the first row, first column right away. The reason why is that we had nice numbers to cancel out some of the other terms in the same column. □

(c) $-2b + 3c = 1$
$3a + 6b - 3c = -2$
$6a + 6b + 3c = 5$

Solution. Write as the augmented matrix and use row operations to solve:

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right) \xrightarrow{-2R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 0 & -6 & 9 & 11 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 11 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 11 \end{array}\right) \xrightarrow{-3R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 8 \end{array}\right)$$

This last equation says $0 = 8$ which is false, so there is no solution.
Author's note: I want to point out here that I didn't try get a 1 in the first row, first column. The reason why is that we had nice numbers to cancel out some of the other terms in the same column and there ended up being no solution. □

2. Let $\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$.

(a) Find $3\mathbf{v_1} + 2\mathbf{v_2} - \mathbf{v_3}$.

Solution.

$$3\mathbf{v_1} + 2\mathbf{v_2} - \mathbf{v_3} = 3\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3-4-3 \\ 0+2+1 \\ 9+0-3 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 6 \end{pmatrix}$$

□

(b) Can $\mathbf{b_1} = \begin{pmatrix} 1 \\ 1 \\ 9 \end{pmatrix}$ be written as a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$? If so, find it.

Solution. Set up a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ and set it equal to $\mathbf{b_1}$:

$$c_1\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 9 \end{pmatrix}$$

Our goal is to find c_1, c_2, c_3. If we can find a solution, then there exists such a linear combination; if there is no solution, then there does not exist such a linear combination. Write as the augmented matrix and use row operations to solve:

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 3 & 0 & 3 & 9 \end{array}\right) \xrightarrow{-3R_1+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 6 & -6 & 6 \end{array}\right) \xrightarrow{-6R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{2R_2+R_1\to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The third column is missing a free variable, and so c_3 is a free variable. We therefore have infinitely many solutions, and so there does exists a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ that equals $\mathbf{b_1}$.
Set $c_3 = c_3$. From the second row, we have that

$$c_2 - c_3 = 1 \Rightarrow c_2 = 1 + c_3.$$

From the first row, we have that

$$c_1 + c_3 = 3 \Rightarrow c_1 = 3 - c_3.$$

Plug these values into the linear combination we wrote at the beginning:

$$(3 - c_3)\begin{pmatrix}1\\0\\3\end{pmatrix} + (1 + c_3)\begin{pmatrix}-2\\1\\0\end{pmatrix} + c_3\begin{pmatrix}3\\-1\\3\end{pmatrix} = \begin{pmatrix}1\\1\\9\end{pmatrix}.$$

□

(c) Can $\mathbf{b_2} = \begin{pmatrix}1\\-4\\5\end{pmatrix}$ be written as a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$? If so, find it.

Solution. Set up a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ and set it equal to $\mathbf{b_2}$:

$$c_1\begin{pmatrix}1\\0\\3\end{pmatrix} + c_2\begin{pmatrix}-2\\1\\0\end{pmatrix} + c_3\begin{pmatrix}3\\-1\\3\end{pmatrix} = \begin{pmatrix}1\\-4\\5\end{pmatrix}$$

Our goal is to find c_1, c_2, c_3. If we can find a solution, then there exists such a linear combination; if there is no solution, then there does not exist such a linear combination. Write as the augmented matrix and use row operations to solve (use the same steps as in part (b)):

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -4 \\ 3 & 0 & 3 & 5 \end{array}\right) \xrightarrow{-3R_1+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 6 & -6 & 2 \end{array}\right) \xrightarrow{-6R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 26 \end{array}\right)$$

The last row tells us that $0 = 26$, which is false. There is no solution, and so there does not exists a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ that equals $\mathbf{b_2}$. □

(d) Are $\mathbf{b_1}$, $\mathbf{b_2}$ in span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$? Use your results from parts (b) and (c).

Solution. Recall that span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is the set of all possible linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. In part (b) we found that we could write $\mathbf{b_1}$ as a linear combination of these vectors, hence $\mathbf{b_1} \in$ span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. In part (c) we found that we could not write $\mathbf{b_2}$ as a linear combination of these vectors, hence $\mathbf{b_2} \notin$ span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. □

(e) Does $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ span $\mathbb{R}^3$?

Solution. We have the following theorem: If A is an $m \times n$ matrix, its columns span $\mathbb{R}^m$ if and only if A has a pivot in each row.

We construct A out of our vectors and apply the same row operations as earlier:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 3 & 0 & 3 \end{pmatrix} \to ... \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We have three rows but only two pivots. There is not a pivot in each row, hence the column of A (which are $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$) do not span $\mathbb{R}^3$. □

3. Let $\mathbf{v_1} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} 0 \\ -3 \\ 8 \end{pmatrix}$, $\mathbf{v_3} = \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}$. Does $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ span $\mathbb{R}^3$? Why or why not?

Solution. We have the following theorem: If A is an $m \times n$ matrix, its columns span $\mathbb{R}^m$ if and only if A has a pivot in each row.

We construct A out of our vectors and apply the same row operations as earlier:

$$A = \begin{pmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -2 & 8 & -5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{pmatrix}$$

We have three rows and three pivots. There is a pivot in each row, hence the column of A (which are $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$) span $\mathbb{R}^3$. □

4. Find the general solution of the system whose augmented matrix is given by

(a) $\left(\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \end{array}\right)$

Solution. Let's write the matrix in reduced row echelon form:

$$\left(\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \end{array}\right) \xrightarrow{3R_2 + R_1 \to R_1} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & -12 & 1 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \end{array}\right)$$

$$\xrightarrow{R_1 + R_3 \to R_1} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \end{array}\right)$$

We have three pivots which correspond with x_1, x_2, x_4. The columns with missing pivots correspond with x_3 and x_5 so they are free variables. Let $x_3 = s$, $x_5 = t$. From the third row we have that

$$x_4 + 9x_5 = 4 \Rightarrow x_4 = 4 - 9x_5 \Rightarrow x_4 = 4 - 9t.$$

From the second row we have that

$$x_2 - 4x_5 = 1 \Rightarrow x_2 = 1 + 4x_5 \Rightarrow x_2 = 1 + 4t.$$

From the first row we have that

$$x_1 - 3x_5 = 5 \Rightarrow x_1 = 5 + 3x_5 \Rightarrow x_1 = 5 + 3t.$$

Therefore the solution to the system is the vector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5+3t \\ 1+4t \\ s \\ 4-9t \\ t \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} 3 \\ 4 \\ 0 \\ -9 \\ 1 \end{pmatrix} t$$

□

(b) $\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$

Solution. Reduce the matrix:

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right) \xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_4 \to R_4} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right) \xrightarrow{5R_2+R_3 \to R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

$$\xrightarrow{4R_2+R_4 \to R_4} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right) \xrightarrow{\frac{1}{6}R_4 \to R_4} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{3R_3+R_2 \to R_2} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{2R_2 + R_1 \to R_1} \left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We have two pivots which correspond with x_1, x_3, x_6. The columns with missing pivots correspond with x_2, x_4, x_5 so they are free variables. Let $x_2 = s$, $x_4 = t$, $x_5 = u$. The fourth row gives us no information. From the third row we have that

$$x_6 = \frac{1}{3}$$

From the second row we have that

$$x_3 + 2x_4 = 0 \Rightarrow x_3 = -2x_4 \Rightarrow x_3 = -2t$$

From the first row we have that

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \Rightarrow x_1 = -3x_2 - 4x_4 - 2x_5 \Rightarrow x_1 = -3s - 4t - 2u$$

Therefore the solution to the system is the vector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3s - 4t - 2u \\ s \\ -2t \\ t \\ u \\ 1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u$$

□

5. Show whether the following sets of vectors in $\mathbb{R}^3$ are linearly independent or linearly dependent.

   (a) $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 10 \\ 2 \end{pmatrix}$

   Solution. We need to look at linear combination formula set to $\mathbf{0}$:

$$c_1 \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

   If $c_1 = c_2 = 0$ then these vectors will be linearly independent; otherwise they are linearly dependent. We can write this equation as the augmented matrix as use row operations to solve:

$$\left(\begin{array}{cc|c} 4 & -4 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array}\right) \xrightarrow{\frac{1}{4}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array}\right) \xrightarrow{R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 9 & 0 \\ 2 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_3\to R_3} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 9 & 0 \\ 0 & 4 & 0 \end{array}\right) \xrightarrow{\frac{1}{9}R_2\to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{array}\right) \xrightarrow{-4R_2+R_3\to R_3} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2+R_1\to R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

This shows that we have the unique solution $c_1 = c_2 = 0$. Thus the vectors are linearly independent.

Another way: You can also use the fact that since we have two vectors that are not multiples of each other then they must be linearly independent. ☐

(b) $\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

Solution. We need to look at linear combination formula set to **0**:

$$c_1 \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $c_1 = c_2 = c_3 = 0$ then these vectors will be linearly independent; otherwise they are linearly dependent. We can write this equation as the augmented matrix as use row operations to solve:

$$\left(\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right) \xrightarrow{R_3+R_1\to R_1} \left(\begin{array}{ccc|c} 1 & 7 & 4 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right) \xrightarrow{R_3-4R_1\to R_3} \left(\begin{array}{ccc|c} 1 & 7 & 4 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -26 & -13 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 7 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -26 & -13 & 0 \end{array}\right) \xrightarrow{26R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 7 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -39 & 0 \end{array}\right) \xrightarrow{-\frac{1}{39}R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 7 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_3+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 7 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-4R_3+R_1\to R_1} \left(\begin{array}{ccc|c} 1 & 7 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-7R_2+R_1\to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This shows the unique solution is $c_1 = c_2 = c_3 = 0$. Thus the vectors are linearly independent. ☐

(c) $\begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

Solution. We need to look at linear combination formula set to **0**:

$$c_1 \begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $c_1 = c_2 = 0$ then these vectors will be linearly independent; otherwise they are linearly dependent. We can write this equation as the augmented matrix as use row operations to solve:

$$\left(\begin{array}{cc|c} 8 & 4 & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 8 & 4 & 0 \\ 3 & 1 & 0 \end{array}\right) \xrightarrow{8R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 3 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{3R_1 + R_3 \to R_3} \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{array}\right) \xrightarrow{-4R_2 + R_3 \to R_3} \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-R_1 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

□

This shows the unique solution is $c_1 = c_2 = 0$. Thus the vectors are linearly independent.

Another way: You can also use the fact that since we have two vectors that are not multiples of each other then they must be linearly independent.

(d) $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix}$

Solution. We need to look at linear combination formula set to $\mathbf{0}$:

$$c_1 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + c_3 \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $c_1 = c_2 = c_3 = c_4 = 0$ then these vectors will be linearly independent; otherwise they are linearly dependent. We can write this equation as the augmented matrix as use row operations to solve:

$$\left(\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -2 & 3 & 6 & 7 & 0 \end{array}\right) \xrightarrow{R_1 + R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 13 & 8 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-6R_2 + R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 14 & 3 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 1 & 14 & 3 & 0 \\ 0 & 2 & -1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_2 + R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 1 & 14 & 3 & 0 \\ 0 & 0 & -29 & -6 & 0 \end{array}\right)$$

□

At this point we notice that the fourth column is missing a pivot, therefore there is a free variable. Since there is a free variable there will be a nontrivial solution, hence the set of vectors is linearly dependent.

Another way: We have a 3×4 matrix, so we will always have less pivots than the number of columns. This implies that we will have a free variable, which gives infinitely many solutions which are not trivial. Therefore the set of vectors is linearly dependent.