Midterm 1 Review

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Summary

This Midterm Review contains notes on sections 1.1−1.5 and 1.7 in your textbook. For your midterm, you should...

- Know how to write a system as an augmented matrix
- Know how to use basic row operations to write a matrix in echelon form
- Know how to use basic row operations to write a matrix in reduced row echelon form
- Know how to use basic row operations to solve a system using echelon or reduced row echelon form
- Know how to identify basic variables and free variables from a matrix in echelon or reduced row echelon form
- Know how to determine if a vector is a linear combination of a given set of vectors
- Know how to determine if a vector is in the span of a given set of vectors
- Know how to determine if a set of vectors spans a space
- Know how to write a matrix equation as an augmented matrix and vice versa
- Know how to write a solution to a system in parametric vector form
- Know how to identify when a set of vectors is linearly independent or linearly dependent

If you are not sure if you know how to do any of the above, you should read the appropriate notes and do some practice problems from your homework and textbook.

1.1 Systems of Linear Equations

A **linear equation** is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

where $x_1, ..., x_n$ are variables, $a_1, ..., a_n$ are constants (these are also called the **coefficients**), and b is also a constant.

Example. Consider the following equations:

(a) $4x + 3z = 1$ (b) $8\sqrt{x_1} - 2x_2 + 4x_3^{-1} = 2$ (c) $x^2 + xy + y^2 = 1$ (d) $x_1 + 2x_2 + 3x_3 - 4x_4 = 0$

The equations in (a) and (d) are linear because they can be written in the form above. The equations (b) and (c) are not linear due to the square root, exponents, and multiplication between variables. □

A **system of linear equations** is a collection of one or more linear equations. A system of linear equations is said to be **consistent** if there is a solution, or is said to be **inconsistent** if there is no solution. A consistent system is said to be **independent** if there is exactly one solution, or is said to be **dependent** if there are infinitely many solutions. To summarize:

- Inconsistent: no solution to the system
- Consistent: there is a solution to the system
  - Independent: exactly one solution to the system
  - Dependent: infinitely many solutions to the system

We can write systems of linear equation as an **augmented matrix**. We then use elementary row operations to solve the matrix/system:

- Interchange any two rows
- Multiply by a row by a nonzero constant (scaling)
- Add multiples of rows to each other and replace one of these rows (replacement)

1.2 Row Reduction and Echelon Forms

A matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row (called the **pivot**) is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry (pivot) are zeros.

I personally like to have my pivots (leading entries) be 1, but your textbook does not indicate this preference. The process of getting to echelon form is called Gaussian elimination.

If a matrix is an echelon form, it is said to be in **reduced echelon form** (or **reduced row echelon form**):

1. The leading entry (pivot) in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

The process of getting to reduced row echelon form is called Gauss-Jordan elimination.

Example. Consider the following matrices

$$(a) \begin{pmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad (b) \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 5 \end{pmatrix} \quad (c) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (d) \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad (e) \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrices (a) and (d) are in row echelon form. The matrices (c) and (e) are in reduced row echelon form. The matrix (b) is neither in echelon nor reduced row echelon form due to the second row. □

If we are solving an augmented matrix, say for example a 3×3 system, then for row echelon our goal will be

$$\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right);$$

and our goal for reduced row echelon will be

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right).$$

The variables that correspond with pivot columns are said to be **basic variables**. The variables that correspond with columns missing pivots are said to be **free variables**. Whenever there is a presence of a free variable the corresponding system will have infinitely many solutions. In some cases, the presence of a free variable will correspond with a row of zeros (this is especially true for square systems, like a 3×3 system). A system will be inconsistent if we get a false equation such as $0 = 2$.

1.3 Vector Equations

A matrix with one column is called a **vector**. A vector with m rows is a vector in the space $\mathbb{R}^m$. Our current operations for vectors right now only consist of addition (including subtraction) and scalar multiplication. We will learn more operations later on.

Let $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ be vectors. We say that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}$$

is a **linear combination** of the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$, where $c_1, c_2, ..., c_n$ are constants.

We define span$\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ to be the set of all linear combinations of the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$. There are infinitely many linear combinations and so span$\{\}$ is an infinite set.

A vector $\mathbf{b} \in \text{span}\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ if and only if we can write $\mathbf{b}$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$. To do this, we can solve the vector equation

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n} = \mathbf{b}$$

for $c_1, c_2, ..., c_n$ by writing the equation as the augmented matrix

$$\begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & ... & \mathbf{v_n} & | & \mathbf{b} \end{pmatrix}$$

and reducing the matrix.

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

If A is an $m \times n$ matrix, with columns $\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$ is given by

$$A\mathbf{x} = \begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & ... & \mathbf{a_n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + ... + x_n\mathbf{a_n}.$$

For $\mathbf{b}$ in $\mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution as the equation

$$x_1\mathbf{a_1} + x_2\mathbf{a_2} + ... + x_n\mathbf{a_n} = \mathbf{b},$$

which also has the same solution as the augmented matrix

$$\begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & ... & \mathbf{a_n} & | & \mathbf{b} \end{pmatrix}.$$

The following is a very useful theorem:

Theorem 1. *Let A be an $m \times n$ matrix. Then the following statements are equivalent (if one is true, all are true; if one is false, all are false):*

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbf{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

A is a matrix of vectors (the vectors are the columns). If we can show that A has a pivot position in every row using row operations, then the vectors that make up A span the space. We can therefore write any matrix in that space as a linear combination of these vectors.

1.5 Solution Sets of Linear Systems

The matrix equation $A\mathbf{x} = \mathbf{0}$ is said to be a **homogeneous** equation. Homogeneous is just a fancy way of saying our equation is equal to "0" (this term will show up in later math courses). $\mathbf{x} = \mathbf{0}$ is certainly a solution to this equation; this is called the **trivial solution**. Some systems have more solutions to this equation, and these are called **nontrivial solutions**. These occur if and only if the equation has at least one free variable.

If a linear system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, the general solution can be written in the **parametric vector form** (a linear combination of vectors that satisfy the equation).

Example. Suppose a system of equations has the solution $x_1 = 4x_3 - 1, x_2 = -x_3 + 3, x_3 = x_3$ (notice here that x_3 is a free variable). We can write our solution as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_3 - 1 \\ -x_3 + 3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}.$$

This right hand side is in parametric form as it is a linear combination of vectors. □

1.7 Linear Independence

A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ are said to be **linearly independent** if the vector equation

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n} = \mathbf{0}$$

has the trivial solution $c_1 = c_2 = ... = c_n = 0$. If $c_1, c_2, ..., c_n$ are not all equal to zero and satisfy the above equation, then the vectors are **linearly dependent**. If the vectors are linearly dependent, we say the equation above is the **linear dependence relation** among $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$.

We can also extend linear independence and dependence to a matrix. Consider the matrix $A = \begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & ... & \mathbf{a_n} \end{pmatrix}$. The columns of a matrix A (remember, columns are vectors) are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.