Solutions to Midterm 2 Practice Problems

Written by Victoria Kala
vtkala@math.ucsb.edu
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Answers

This page contains answers only. Detailed solutions are on the following pages.

1. (a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix}$
   (b) $T(2,-1) = (-1,-2,-1,3)$
   (c) $ker(T) = \{\mathbf{0}\}$
   (d) $range(T) = \text{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ -1 \end{pmatrix} \right\}$

2. (a) Linear
   (b) Not linear
   (c) Linear
   (d) Not linear

3. (a) $\begin{pmatrix} 9 & -4 \\ 0 & 1 \end{pmatrix}$
   (b) $\begin{pmatrix} 8 & -7 & 1 \\ -3 & 5 & -2 \end{pmatrix}$
   (c) Undefined
   (d) $\begin{pmatrix} 9 & 27 & 42 \\ 31 & 41 & 40 \\ 20 & 34 & 41 \end{pmatrix}$
   (e) Undefined
   (f) $\frac{1}{8^3}\begin{pmatrix} 8 & 28 \\ 0 & 64 \end{pmatrix}$
   (g) $\begin{pmatrix} 21 & 17 \\ 17 & 35 \end{pmatrix}$

4. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

5. (a) $\det(G) = 0$
   (b) G^{-1} does not exist

6. (a) $\det(J) = 105$
   (b) $J^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1/3 & 0 & 0 \\ 0 & -1/5 & 1/5 & 0 \\ 0 & 0 & -1/7 & 1/7 \end{pmatrix}$

7. U is not a subspace

8. See detailed solution

9. (a) $col(K) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 9 \\ -4 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 9 \\ -5 \end{pmatrix} \right\}$
   (b) $null(K) = \text{span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 14 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 37 \\ 0 \\ -4 \\ 0 \\ -5 \\ 1 \end{pmatrix} \right\}$

Detailed Solutions

1. Let T be the linear transformation defined by the formula

$$T(x_1, x_2) = (x_2, -x_1, x_1 + 3x_2, x_1 - x_2).$$

(a) Find the standard matrix A for the linear transformation such that $T(\mathbf{x}) = A\mathbf{x}$.

Solution. Recall that the standard matrix A is given by $A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix}$ where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. We need to find $T(1, 0)$ and $T(0, 1)$. Using the formula above, we see that

$$T(1, 0) = (0, -1, 1, 1) \quad \text{and} \quad T(0, 1) = (1, 0, 3, -1).$$

Therefore $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix}$.

Let's check:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix}$$

This is the same as $T(x_1, x_2)$! □

(b) Find the image of $(x_1, x_2) = (2, -1)$.

Solution. Use the formula above to find $T(2, -1)$ (or you can use matrix multiplication using the matrix in (a)):

$$T(2, -1) = (-1, -2, -1, 3).$$

□

(c) Find the kernel of T (*Hint*: This is the null space of A).

Solution. The kernel of T is the set of all vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$. But, since $T(\mathbf{x}) = A\mathbf{x}$, then we solve $A\mathbf{x} = \mathbf{0}$ (this is the null space of A):

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{R2+R3\to R3} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{R2+R4\to R4} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-3R1+R3\to R3} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right) \xrightarrow{R1+R4\to R4} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R2 \leftrightarrow R1} \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{-R1 \to R1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

□

This shows $x_1 = x_2 = 0$, hence $\mathbf{x} = \mathbf{0}$. This is the only solution to $A\mathbf{x} = \mathbf{0}$, hence

$$ker(T) = null(A) = \{\mathbf{0}\}.$$

(d) Find the range of T (*Hint*: This is the column space of A).

Solution. The range of T is the column space of A. The column space of A is the span of the columns:

$$col(A) = \text{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ -1 \end{pmatrix} \right\}.$$

This is the solution.

However, here is another question: is this a basis of the column space? The answer is yes! We found the reduced row echelon form of A to be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. The first and second column each have a pivot, hence the first and second column of A are linearly independent and span the space. □

2. Determine whether $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator where

(a) $T(x, y) = (2x + y, x - y)$

Solution. Linear operator means linear transformation. For $(a) - (d)$ we need to show that:

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ z \end{pmatrix}\right) = T\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} w \\ z \end{pmatrix}$$

and

$$T\left(c\begin{pmatrix} x \\ y \end{pmatrix}\right) = cT\begin{pmatrix} x \\ y \end{pmatrix}.$$

Let's look at addition:

$$T\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 2x + y \\ x - y \end{pmatrix} + \begin{pmatrix} 2w + z \\ w - z \end{pmatrix} = \begin{pmatrix} 2x + y + 2w + z \\ x - y + w - z \end{pmatrix}$$

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ z \end{pmatrix}\right) = T\begin{pmatrix} x + w \\ y + z \end{pmatrix} = \begin{pmatrix} 2(x + w) + (y + z) \\ (x + w) - (y + z) \end{pmatrix}$$

We see that $T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ z \end{pmatrix}\right) = T\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} w \\ z \end{pmatrix}$, and so the first property holds.

Let's look at scalar multiplication:

$$T\left(c\begin{pmatrix}x\\y\end{pmatrix}\right) = T\begin{pmatrix}cx\\cy\end{pmatrix} = \begin{pmatrix}2cx+cy\\cx-cy\end{pmatrix}$$
$$cT\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = c\begin{pmatrix}2x+y\\x-y\end{pmatrix} = \begin{pmatrix}c(2x+y)\\c(x-y)\end{pmatrix}$$

We see that $T\left(c\begin{pmatrix}x\\y\end{pmatrix}\right) = cT\begin{pmatrix}x\\y\end{pmatrix}$, and so the second property holds.

Thus T is a linear transformation. □

(b) $T(x,y) = (x+1,y)$

Solution. Let's look at addition:

$$T\begin{pmatrix}x\\y\end{pmatrix} + T\begin{pmatrix}w\\z\end{pmatrix} = \begin{pmatrix}x+1\\y\end{pmatrix} + \begin{pmatrix}w+1\\z\end{pmatrix} = \begin{pmatrix}x+w+2\\y+z\end{pmatrix}$$
$$T\left(\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}w\\z\end{pmatrix}\right) = T\begin{pmatrix}x+w\\y+z\end{pmatrix} = \begin{pmatrix}x+w+1\\y+z\end{pmatrix}$$

Therefore $T\left(\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}w\\z\end{pmatrix}\right) \neq T\begin{pmatrix}x\\y\end{pmatrix} + T\begin{pmatrix}w\\z\end{pmatrix}$, and so the first property doesn't hold.
This is enough to show that the transformation is not linear.
It also doesn't hold for scalar multiplication:

$$T\left(c\begin{pmatrix}x\\y\end{pmatrix}\right) = T\begin{pmatrix}cx\\cy\end{pmatrix} = \begin{pmatrix}cx+1\\cy\end{pmatrix}$$
$$cT\begin{pmatrix}x\\y\end{pmatrix} = c\begin{pmatrix}x+1\\y\end{pmatrix} = \begin{pmatrix}cx+c\\cy\end{pmatrix}$$

Therefore $T\left(c\begin{pmatrix}x\\y\end{pmatrix}\right) \neq cT\begin{pmatrix}x\\y\end{pmatrix}$, and so the second property doesn't hold.

Since neither property holds then T is not a linear transformation. (Note: you only need to show that one property fails, so choose whichever one seems easiest to you.) □

(c) $T(x,y) = (y,y)$

Solution. Let's look at addition:

$$T\begin{pmatrix}x\\y\end{pmatrix} + T\begin{pmatrix}w\\z\end{pmatrix} = \begin{pmatrix}y\\y\end{pmatrix} + \begin{pmatrix}z\\z\end{pmatrix} = \begin{pmatrix}y+z\\y+z\end{pmatrix}$$
$$T\left(\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}w\\z\end{pmatrix}\right) = T\begin{pmatrix}x+w\\y+z\end{pmatrix} = \begin{pmatrix}y+z\\y+z\end{pmatrix}$$

We see that $T\left(\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}w\\z\end{pmatrix}\right) = T\begin{pmatrix}x\\y\end{pmatrix} + T\begin{pmatrix}w\\z\end{pmatrix}$, and so the first property holds.

Let's look at scalar multiplication:

$$T\left(c\begin{pmatrix}x\\y\end{pmatrix}\right) = T\begin{pmatrix}cx\\cy\end{pmatrix} = \begin{pmatrix}cy\\cy\end{pmatrix}$$
$$cT\begin{pmatrix}x\\y\end{pmatrix} = c\begin{pmatrix}y\\y\end{pmatrix} = \begin{pmatrix}cy\\cy\end{pmatrix}$$

We see that $T\left(c\begin{pmatrix}x\\y\end{pmatrix}\right) = cT\begin{pmatrix}x\\y\end{pmatrix}$, and so the second property holds.
Thus T is a linear transformation. □

(d) $T(x, y) = (\sqrt[3]{x}, \sqrt[3]{y})$

Solution. Let's look at addition:

$$T\begin{pmatrix}x\\y\end{pmatrix} + T\begin{pmatrix}w\\z\end{pmatrix} = \begin{pmatrix}\sqrt[3]{x}\\\sqrt[3]{y}\end{pmatrix} + \begin{pmatrix}\sqrt[3]{w}\\\sqrt[3]{z}\end{pmatrix} = \begin{pmatrix}\sqrt[3]{x}+\sqrt[3]{w}\\\sqrt[3]{y}+\sqrt[3]{z}\end{pmatrix}$$
$$T\left(\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}w\\z\end{pmatrix}\right) = T\begin{pmatrix}x+w\\y+z\end{pmatrix} = \begin{pmatrix}\sqrt[3]{x+w}\\\sqrt[3]{y+z}\end{pmatrix}$$

Therefore $T\left(\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}w\\z\end{pmatrix}\right) \neq T\begin{pmatrix}x\\y\end{pmatrix} + T\begin{pmatrix}w\\z\end{pmatrix}$, and so the first property doesn't hold.
This is enough to show that the transformation is not linear.
It also doesn't hold for scalar multiplication:

$$T\left(c\begin{pmatrix}x\\y\end{pmatrix}\right) = T\begin{pmatrix}cx\\cy\end{pmatrix} = \begin{pmatrix}\sqrt[3]{cx}\\\sqrt[3]{cy}\end{pmatrix}$$
$$cT\begin{pmatrix}x\\y\end{pmatrix} = c\begin{pmatrix}\sqrt[3]{x}\\\sqrt[3]{y}\end{pmatrix} = \begin{pmatrix}c\sqrt[3]{x}\\c\sqrt[3]{y}\end{pmatrix}$$

Therefore $T\left(c\begin{pmatrix}x\\y\end{pmatrix}\right) \neq cT\begin{pmatrix}x\\y\end{pmatrix}$, and so the second property doesn't hold.
Since neither property holds then T is not a linear transformation. (Note: you only need to show that one property fails, so choose whichever one seems easiest to you.) □

3. Consider the matrices

$$A = \begin{pmatrix}3 & 0\\-1 & 2\\1 & 1\end{pmatrix}, \quad B = \begin{pmatrix}4 & -1\\0 & 2\end{pmatrix}, \quad C = \begin{pmatrix}1 & 4 & 2\\3 & 1 & 5\end{pmatrix}, \quad D = \begin{pmatrix}1 & 5 & 2\\-1 & 0 & 1\\3 & 2 & 4\end{pmatrix}$$

Compute the following (where possible). If the operation is not defined, explain why.

(a) $B^2 - 2B + I$.

Solution. B is a 2×2 matrix. B^2 will also be a 2×2 matrix, $2B$ will be a 2×2 matrix, and I will be a 2×2 matrix, therefore $B^2 - 2B + I$ is defined, and

$$\begin{aligned} B^2 - 2B + I &= \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix} - 2 \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 16 & -6 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} -8 & 2 \\ 0 & -4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 9 & -4 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

□

(b) $3A^T - C$

Solution. A is a 3×2 matrix, and so A^T will be a 2×3 matrix. C is also 2×3 matrix, hence $3A^T - C$ is defined, and

$$\begin{aligned} 3A^T - C &= 3 \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}^T - \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \\ &= 3 \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 3 \cdot 3 - 1 & 3 \cdot (-1) - 4 & 3 \cdot 1 - 2 \\ 3 \cdot 0 - 3 & 3 \cdot 2 - 1 & 3 \cdot 1 - 5 \end{pmatrix} \\ &= \begin{pmatrix} 8 & -7 & 1 \\ -3 & 5 & -2 \end{pmatrix} \end{aligned}$$

□

(c) BD

Solution. B is a 2×2 matrix, D is a 3×3 matrix. The number columns of B are not the same as the number rows of D, hence BD is not defined. □

(d) $(AC)D$

Solution. A is a 3×2 matrix, C is a 2×3 matrix. AC is defined and will be a 3×3

matrix. D is a 3×3 matrix, and so $(AC)D$ is defined.

$$
\begin{aligned}
(AC)D &= \left(\begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \right) \begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 3+0 & 12+0 & 6+0 \\ -1+6 & -4+2 & -2+10 \\ 1+3 & 4+1 & 2+5 \end{pmatrix} \begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 12 & 6 \\ 5 & -2 & 8 \\ 4 & 5 & 7 \end{pmatrix} \begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 3-12+18 & 15+0+12 & 6+12+24 \\ 5+2+24 & 25+0+16 & 10-2+32 \\ 4-5+21 & 20+0+14 & 8+5+28 \end{pmatrix} \\
&= \begin{pmatrix} 9 & 27 & 42 \\ 31 & 41 & 40 \\ 20 & 34 & 41 \end{pmatrix}
\end{aligned}
$$

□

(e) $CB - 2A$

Solution. C is a 2×3 matrix, B is a 2×2 matrix. The number of columns of C is not the same as the number of rows of B, so CB is undefined. Therefore $CB - 2A$ is not defined. □

(f) B^{-3}

Solution. $B^{-3} = (B^{-1})^3$. B^{-1} will be a 2×2 matrix, and will be defined since $\det(B) = 4 \cdot 2 - 0 = 8 \neq 0$, and

$$B^{-1} = \frac{1}{8} \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}.$$

$(B^{-1})^3$ will be a 2×2 matrix and

$$
\begin{aligned}
(B^{-1})^3 &= \left(\frac{1}{8} \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix} \right)^3 \\
&= \frac{1}{8^3} \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix} \\
&= \frac{1}{8^3} \begin{pmatrix} 4 & 6 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix} \\
&= \frac{1}{8^3} \begin{pmatrix} 8 & 28 \\ 0 & 64 \end{pmatrix}
\end{aligned}
$$

□

(g) CC^T

Solution. C is a 2×3 matrix and C^T is a 3×2 matrix. The number of columns of C match the number of rows of C^T, so CC^T is defined, and

$$\begin{aligned}
\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}^T &= \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 1+16+4 & 3+4+10 \\ 3+4+10 & 9+1+25 \end{pmatrix} \\
&= \begin{pmatrix} 21 & 17 \\ 17 & 35 \end{pmatrix}
\end{aligned}$$

□

4. Find the inverse of $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Solution. The inverse is well defined since

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

and

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} = \frac{1}{1} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Let's check our solution:

$$\begin{aligned}
\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

The other direction works as well. □

5. Let $G = \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{pmatrix}$.

(a) Find $\det(G)$.

Solution. Let's use cofactor expansion by crossing out the first column:

$$\begin{aligned}
\begin{vmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{vmatrix} &= 1 \begin{vmatrix} 3 & 4 \\ 6 & 0 \end{vmatrix} - 0 \begin{vmatrix} -5 & -4 \\ 6 & 0 \end{vmatrix} + (-3) \begin{vmatrix} -5 & -4 \\ 3 & 4 \end{vmatrix} \\
&= 1(0-24) - 0 - 3(-20+12) \\
&= -24 + 24 \\
&= 0
\end{aligned}$$

Thus $\det(G) = 0$. ☐

(b) Does G^{-1} exist? If so, find it.

Solution. No, G^{-1} does not exist since $\det(G) = 0$. ☐

6. Let $J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{pmatrix}$.

(a) Find $\det(J)$.

Solution. J is a lower triangular matrix, hence the determinant of J is the product of the diagonal entries:

$$\det(J) = 1 \cdot 3 \cdot 5 \cdot 7 = 105.$$

☐

(b) Does J^{-1} exist? If so, find it.

Proof. We set up the augmented matrix $\begin{pmatrix} J & I \end{pmatrix}$ and reduce to get $\begin{pmatrix} I & J^{-1} \end{pmatrix}$:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R2-R1\to R2} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R3-R1\to R3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R4-R1\to R4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R3-R2\to R3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R4-R2\to R4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R4-R3\to R4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & -1 & 1 \end{array}\right) \xrightarrow{\frac{1}{3}R2\to R2} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R3\to R3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/5 & 1/5 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R3 \to R3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/5 & 1/5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/7 & 1/7 \end{array}\right)$$

□

Therefore $J^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1/3 & 0 & 0 \\ 0 & -1/5 & 1/5 & 0 \\ 0 & 0 & -1/7 & 1/7 \end{pmatrix}$.

Let's check our solution:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1/3 & 0 & 0 \\ 0 & -1/5 & 1/5 & 0 \\ 0 & 0 & -1/7 & 1/7 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/3+1/3 & 3/3 & 0 & 0 \\ -1/5+1/5 & -3/5+3/5 & 5/5 & 0 \\ -1/7+1/7 & -3/7+3/7 & -5/7+5/7 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The other direction works as well.

7. Let $U = \{(x, y) : x \geq -2, y \leq 1\}$ be a subset of $\mathbb{R}^2$. Is U a subspace of $\mathbb{R}^2$? Why or why not?

Solution. You should always sketch the subset whenever possible. U can easily be sketched out (see the figure below, U is the dark shaded set).

We need to verify look at the three properties of a subspace to see if U is a subspace:

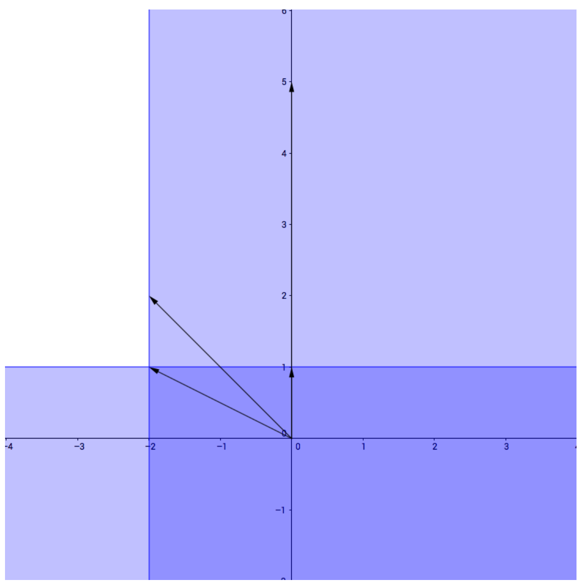
(i) The zero vector of $\mathbb{R}^2$ is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in U$ since $0 \geq -2$ and $0 \leq 1$.

If you are looking at this graphically, you can clearly see that the zero vector is in the set U.

(ii) We need to pick two vectors in U and add them together. Let $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \in U$. Then $x \geq -2$, $w \geq -2$, $y \leq 1$, and $z \leq 1$. When we add the two vectors together we have $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} x+w \\ y+z \end{pmatrix}$. The inequalities will also add together, and we will have $x + w \geq -4$ and $y + z \leq 2$. Since these inequalities are not preserved (they aren't ≥ -2 and ≤ 1), then $\begin{pmatrix} x+w \\ y+z \end{pmatrix} \notin U$, hence U is not closed under addition.

If you are looking at this graphically, you need to find two vectors that when added together they are no longer in the set U. One such example would be the vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, which are both in U. But their sum is $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$, which is not in U. (These vectors are plotted in the figure below.)

(iii) We need to pick a vector in U and multiply it by a scalar. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in U$ and $c \in \mathbb{R}$. Then $x \geq -2$ and $y \leq 1$. If we multiply c by our vector, we have $\begin{pmatrix} cx \\ cy \end{pmatrix}$. If $c \geq 0$ then our inequalities will be $cx \geq -2c$ and $cy \leq c$; if $c < 0$ then our inequalities will be $cx \leq -2c$ and $cy \geq c$. Since these inequalities are not preserved, then U is not closed under scalar multiplication.
If you are looking at this graphically, you need to find a vector and a scalar such that a scalar multiplied by this vector will no longer be in the set U. One such example would be the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the scalar 5: $5\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \notin U$. (These vectors are plotted in the figure below.)



Above we showed that U fails under addition and scalar multiplication, therefore U is not a subspace. Showing just one of these fails is enough to show that U is not a subspace. □

8. Let $\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0), \mathbf{v}_3 = (3, 3, 4)$. Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Solution. To show that S is a basis we need to show that (i) all the vectors in S are linearly independent and (ii) the vectors in S span the space, which in this case is $\mathbb{R}^3$.

(i) Linearly Independent: We need to take an arbitrary linear combination and set it equal to the zero vector:
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

We set up an augmented matrix with our vectors as the columns and reduce to solve for c_1, c_2, c_3:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 9 & 3 & 0 \\ 1 & 0 & 4 & 0 \end{array}\right) \xrightarrow{-2R1+R2\to R2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & -3 & 0 \\ 1 & 0 & 4 & 0 \end{array}\right) \xrightarrow{-R1+R3\to R3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{2R3+R2\to R2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right) \xrightarrow{2R2+R3\to R3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-R3\to R3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This is enough for us to solve the system. This shows that $c_3 = c_2 = c_1 = 0$. This proves that the set of vectors in S are linearly independent.

(ii) Span: In the previous part we showed that the matrix formed by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ has a pivot in every row. This implies that these vectors span $\mathbb{R}^3$.

You can also use the fact that a set of n linearly independent vectors spans $\mathbb{R}^n$.

□

9. Let $K = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix}$.

(a) Find a basis for the column space of K.

Solution. The column space of K is the spanning set of all the columns of K:

$$col(K) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \\ -6 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 9 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 9 \\ -5 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 7 \\ -4 \end{pmatrix} \right\}$$

However, we are asked to find the **basis** of $col(K)$, which means we only want to find the linearly independent vectors in this set. We will do this by reducing K:

$$\begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix} \xrightarrow{-2R1+R2\to R2} \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix}$$

$$\xrightarrow{-2R1+R3\to R3} \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix} \xrightarrow{R1+R3\to R1} \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-R2+R3\to R3} \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{2R3+R2\to R2} \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-5R3+R1\to R1} \begin{pmatrix} 1 & -3 & 4 & -2 & 0 & -21 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-4R2+R1\to R1} \begin{pmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The columns with pivots are the 1st, 3rd, and 5th columns. This means that the 1st, 3rd, and 5th columns of K will form the basis of the column space of K:

$$col(K) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 9 \\ -4 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 9 \\ -5 \end{pmatrix} \right\}$$

□

(b) Find a basis for the null space of K.

Solution. We need to find all the vectors $\mathbf{x}$ such that $K\mathbf{x} = \mathbf{0}$. We do this by reducing the augmented matrix $\begin{pmatrix} K & \mathbf{0} \end{pmatrix}$. We already reduced the matrix in part (a):

$$\left(\begin{array}{cccccc|c} 1 & -3 & 0 & -14 & 0 & -37 & 0 \\ 0 & 0 & 1 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

□

The columns with pivots are associated with x_1, x_3, x_5. The free variables are x_2, x_4, x_6. We set $x_2 = s, x_2 = t, x_3 = u$ for $s, t, u \in \mathbb{R}$. We now solve for x_1, x_3, x_5 in terms of the free variables. From the third row, we have:

$$x_5 + 5x_6 = 0 \quad \Rightarrow \quad x_5 = -5u$$

From the second row we have:

$$x_3 + 3x_4 + 4x_6 = 0 \quad \Rightarrow \quad x_3 = -3t - 4u$$

From the first row we have:

$$x_1 - 3x_2 - 14x_4 - 37x_6 \quad \Rightarrow \quad x_1 = 3s + 14t + 37u$$

We write our solution out in parametric form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3s + 14t + 37u \\ s \\ -3t - 4u \\ t \\ -5u \\ u \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} 14 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 37 \\ 0 \\ -4 \\ 0 \\ -5 \\ 1 \end{pmatrix} u$$

This is the spanning set for null space of K:

$$null(K) = \text{span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 14 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 37 \\ 0 \\ -4 \\ 0 \\ -5 \\ 1 \end{pmatrix} \right\}$$