Midterm 2 Review

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Last Updated 11/08/2015

Summary

This midterm review contains notes on sections 1.8, 1.9, 2.1, 2.2, 2.3, 3.1, 3.2, 4.1, 4.2, 4.3. For your midterm, you should...

- Know how to determine if a transformation is linear
- Know how to find a matrix representation of a linear transformation
- Know how to find a transformation of a vector using linear combinations
- Know how to use matrix operations (addition, scalar multiplication, matrix multiplication, transpose)
- Know how to find an inverse of a matrix
- Know how to represent a row operation as an elementary matrix
- Know how to use elementary matrices to find the inverse of a matrix
- Know how to find a determinant of a matrix using cofactor expansion
- Know how to find a determinant of a matrix using row operations
- Know how to determine if a subset of a vector space is a subspace
- Know how to determine if a set of vectors is a basis of a space
- Know how to find the basis of a null space and column space of a given matrix
- Know how to find the kernel and range of a linear transformation

If you are not sure if you know how to do any of the above, you should read the appropriate notes and do some practice problems from your homework and textbook.

Linear Transformations

See sections 1.8, 1.9 of your textbook.

A **transformation** (also called a function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns a vector $\mathbf{x} \in \mathbb{R}^n$ to a vector $T(\mathbf{x}) \in \mathbb{R}^m$. We say a transformation T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $c \in \mathbb{R}, \mathbf{u}$ in the domain of T.

These properties imply that $T(\mathbf{0}) = \mathbf{0}$ and $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$. Linear combinations are also preserved; that is:

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + ... + c_nT(\mathbf{v}_n).$$

A matrix transformation from $\mathbb{R}^n \to \mathbb{R}^m$ is given by $T(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix. Matrix transformations are linear transformations! The standard matrix representation of A is given by

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & ... & T(\mathbf{e}_n) \end{pmatrix}$$

where $\mathbf{e}_1 = (1, 0, ..., 0), \mathbf{e}_2 = (0, 1, ..., 0), ..., \mathbf{e}_n = (0, 0, ..., 1)$.

Matrix Operations

See section 2.1 of your textbook.

Recall that the size of a matrix is the number of rows by the number of columns. For example, a 3×100 matrix has 3 rows, 100 columns.

We can perform the following operations with matrices:

- Addition: The sum $A + B$ is defined as long as A and B are the same size, and each entry of $A + B$ is the sum of the corresponding entries in A and B.
- Scalar multiplication: If $c \in \mathbb{R}$ and A is a matrix, then cA is the matrix whose entries are all multiplied by c.
- Matrix Multiplication: The product AB is defined as long the number of columns of A is the same as the number or rows of B. It is helpful to remember the following:

  $$\underbrace{A}_{m\times n} \times \underbrace{B}_{n\times p} = \underbrace{C}_{m\times p}$$

  The ijth entry (that is, the ith row, jth column) of AB is found by multiplying the ith row of A and the jth column of B.
- Transpose: The transpose of a matrix A (denoted by A^T) is found by switching its rows and columns.

Example. Let A be the 3×4 matrix is given by

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 5 & -7 & 0 & 0 \end{pmatrix}.$$

The transpose of A is given by

$$A^T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 5 & -7 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 1 & -7 \\ 3 & -1 & 0 \\ 4 & 2 & 0 \end{pmatrix}.$$

Notice that A^T is a 4×3 matrix. (The transpose "swaps" the size of the matrix.)

Some items to remember:

- In general, $AB \neq BA$.
- The cancellation law doesn't always hold; that is, $AB = AC$ does NOT always imply $B = C$.
- $(AB)^T = B^T A^T$

The Inverse of a Matrix

See sections 2.2, 2.3 of your textbook.

An $n \times n$ matrix A is **invertible** (sometimes called **nonsingular**) if there exists an A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

Not all matrices have inverses, however. One way we can check to see if an inverse is invertible is to calculate the determinant (see next section). If $\det A \neq 0$ then A^{-1} exists.

We can use the inverse to solve systems of equations. If A is invertible then the equation $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Some nice properties of inverses:

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

We can easily find the inverse of a 2×2 matrix. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

provided that $ad - bc \neq 0$.

For larger matrices, we use row operations to find inverses. One way is to use **elementary matrices**. An elementary matrix is the matrix representation of a single row operation and is found by applying a row operation to the identity matrix. For example, if we performed the row operation $-4R1 + R3 \to R3$, then the corresponding elementary matrix would be

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}.$$

If we can find elementary matrices $E_1, ..., E_n$ such that $E_n \cdots E_1 A = I$, then $A^{-1} = E_n \cdots E_1$.

An alternate way is to augment our matrix with the identity matrix:

$$\begin{pmatrix} A & I \end{pmatrix}$$

and then use row operations to reduce to the matrix to

$$\begin{pmatrix} I & A^{-1} \end{pmatrix}.$$

Determinants

See sections 3.1, 3.2 of your textbook.

One way to find the determinant of a matrix is to use cofactor expansion. We use the following steps:

1. Write down the corresponding sign matrix (checkerboard of $+$ and $-$, starting with $+$).
2. Choose to cross out a row or column (try to pick the one with the most 0's).
3. If you cross out a row, cross out the columns one by one; if you cross out a column, cross out the rows one by one.
4. Calculate the determinant of the remaining matrices. (Note: you may need to repeat the previous steps to do this.)

There is a gnarly formula for this in the textbook, but I think it is better to just look at examples.

Another way is to use row reduction. We can use row operations to write a matrix in row-echelon form (usually upper triangular). The determinant of a triangular matrix is the product of its diagonal entries. The following are the row reduction rules:

1. If you multiply a row by a number k, then you must multiply the determinant by $\frac{1}{k}$.
2. If two rows are interchanged, the determinant changes sign.
3. If a multiple of one row is added to another, there is no change.

A note about Rule (1): I am talking about this as if you are REDUCING the matrix and writing down your row operations as you go. Your book writes this in an alternative way by saying if a row is multiplied by a number then the determinant is multiplied by that number. These are equivalent definitions, so use whichever one you feel more comfortable with.

When to use cofactor expansion vs. row reduction (if not specified in the directions):

- If you are given a matrix with a lot of 0's, use cofactor expansion.
- If you are given a matrix where rows appear to cancel out nicely, use row reduction.
- If you are given a determinant of a matrix and asked to find the determinant of a rearrangement of that matrix, use row reduction.

Otherwise, just use the method you are most comfortable with.

Vector Spaces and Subspaces

See section 4.1 of your textbook.

A **vector space** is a nonempty set V of objects, called vectors, on which addition and scalar multiplication is defined and has the following properties:

1. If $\mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} \in V$. (We call this "closed under addition".)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u} \in V$ there is a $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. For each scalar $c \in \mathbb{R}$ and $\mathbf{u} \in V$, $c\mathbf{u} \in V$. (We call this "closed under scalar multiplication.")
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1 \cdot \mathbf{u} = \mathbf{u}$

$\mathbb{R}^n$ is a vector space. Other vector spaces include polynomials with real coefficients, continuous functions, and so on.

A **subspace** of a vector space V is a subset $U \subset V$ with the following properties:

1. $\mathbf{0} \in U$ (note that this is the zero vector of the vector space V)
2. If $\mathbf{u}, \mathbf{v} \in U$ then $\mathbf{u} + \mathbf{v} \in U$. (We call this "closed under addition".)

3. For each scalar $c \in \mathbb{R}$ and $\mathbf{u} \in U$, $c\mathbf{u} \in U$. (We call this "closed under scalar multiplication.")

A subspace is a vector space! Once these three properties are verified, the other properties will follow.

If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \in V$, then span$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a subspace of V.

Basis

See section 4.3 of your textbook.

Let V be any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be a set of vectors in V. We say that S is a basis for V if the following conditions hold:

(i) S is linearly independent.

(ii) S spans V.

Example. Let $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$. The set $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ forms what we call the **standard basis** of $\mathbb{R}^3$. Notice that the set S is linearly independent and this set of vectors spans $\mathbb{R}^3$. (See Midterm 1 Review for information on linear independence and span.)

The Null Space and Column Space

See section 4.2 of your textbook.

The **null space** of an $m \times n$ matrix A is the set of all solutions to the equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$Null(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

The null space is a subspace of $\mathbb{R}^n$. To show if a vector $\mathbf{v}$ is in the null space of a matrix A, show that $A\mathbf{v} = \mathbf{0}$. If you are asked to find a spanning set for a null space of a matrix A, find the general solution of $A\mathbf{x} = \mathbf{0}$.

The **column space** of an $m \times n$ matrix A is the set of all linear combinations of the columns of A. In set notation,

$$Col(A) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$$

where $A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & ... & \mathbf{v}_n \end{pmatrix}$. The column space forms a subspace of $\mathbb{R}^m$. To show if a vector $\mathbf{v}$ is in the column space of a matrix A, show that $A\mathbf{x} = \mathbf{v}$ has a solution.

The **kernel** and **range** of a linear transformation $T(\mathbf{x}) = A\mathbf{x}$ are just the null space and column of space of A, respectively.