Math 4A: Quiz 5 Solutions

November 21, 2015

1. Find the coordinate vector of $\mathbf{p}$ relative to the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where

   (a) $\mathbf{p} = 4 - 3x + x^2$; $\mathbf{p_1} = 1, \mathbf{p_2} = x, \mathbf{p}_3 = x^2$

   Solution. We can represent the polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ as the vector (a_0, a_1, a_2). We want to write $\mathbf{p}$ as a linear combination of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$:

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

   By observation we see that $c_1 = 4, c_2 = -3, c_3 = 1$. Therefore $\mathbf{p}_s = (4, -3, 1)$. □

   (b) $\mathbf{p} = 2 - x + x^2$; $\mathbf{p}_1 = 1 + x, \mathbf{p_2} = 1 + x^2, \mathbf{p_3} = x + x^2$

   Solution. We want to write $\mathbf{p}$ as a linear combination of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$:

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

   We can write this as an augmented matrix and use row reduction to solve:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{-R1+R2\to R2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{R2+R3\to R3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 2 & -2 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R3\to R3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{R2-R3\to R2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{-R2\to R2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{R1-R2\to R1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

   □

   Thus $c_1 = 0, c_2 = 2, c_3 = -1$. Therefore $\mathbf{p}_s = (0, 2, -1)$.

2. In each part, use the information in the table to find the dimension of the row space, column space, nullspace of A, and nullspace of A^T:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	9×5	4×4	6×2
Rank(A)	3	2	1	2	2	0	2

Solution. If A is an $m\times n$ matrix and has rank r, then the dimension of $col(A) = r$, dimension of $row(A) = r$, dimension of $null(A) = n - r$, and dimension of $null(A^T) = m - r$. See the following table.

dim	(a)	(b)	(c)	(d)	(e)	(f)	(g)
$col(A)$	3	2	1	2	2	0	2
$row(A)$	3	2	1	2	2	0	2
$null(A)$	0	1	2	8	3	4	0
$null(A^T))$	0	1	2	3	7	4	4

□