Math 4A: Quiz 6 Solutions

December 1, 2015

1. In this quiz we will be calculating A^{13} where

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

The steps are are listed below.

Step 1: Find the eigenvalues of A.

Solution. Find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)\left[-\lambda(3-\lambda) - (-2)(1)\right]$$
$$= (2-\lambda)[\lambda^2 - 3\lambda + 2] = (2-\lambda)(\lambda-2)(\lambda-1)$$
$$= -(\lambda-2)^2(\lambda-1)$$

Set the characteristic polynomial equal to 0:

$$-(\lambda-2)^2(\lambda-1) = 0 \quad \Rightarrow \quad \lambda_1 = 2 \text{ mult. } 2, \lambda_2 = 1 \text{ mult. } 1$$

□

Step 2: Find the eigenvectors of A.

Solution. Use the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to solve for the eigenvectors.
When $\lambda_1 = 2$, we have the system

$$\left(\begin{array}{ccc|c} -\lambda_1 & 0 & -2 & 0 \\ 1 & 2-\lambda_1 & 1 & 0 \\ 1 & 0 & 3-\lambda_1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \to \ldots \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We need to solve for the general solution. x_1 is a pivot, x_2 and x_3 are free variables. The first equation shows that $x_1 = -x_3$. Therefore the general solution to this system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} x_3.$$

This shows that for $\lambda_1 = 2$, we have TWO eigenvectors, and they are

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

When $\lambda_2 = 1$, we have the system

$$\left(\begin{array}{ccc|c} -\lambda_2 & 0 & -2 & 0 \\ 1 & 2-\lambda_2 & 1 & 0 \\ 1 & 0 & 3-\lambda_2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \to \ldots \to \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_1 and x_2 are pivots, x_3 is a free variable. The first equation tells us that $x_1 = -2x_3$, the second equation tells us that $x_2 = x_3$. Therefore the general solution to this system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} x_3.$$

Thus the eigenvector corresponding with $\lambda_2 = 1$ is

$$\mathbf{v}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

□

Step 3: Use your results from the previous steps form the matrix P of eigenvectors and the diagonal matrix D of eigenvalues. *Caution*: Watch out for the ordering!

Solution. You can either form P or D first, it's up to you. I am going to form P first:

$$P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Since λ_1 corresponds with the first two columns of P, it must be in the first two columns of D; since λ_2 corresponds with the last vector of P, it must be in the last column of D:

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Your P and D might look different from mine and that's okay! Just be careful with your ordering.

Also notice that since A is a 3×3 matrix, P and D should also be 3×3 matrices.

□

Step 4: Find P^{-1}.

Solution. Augment P with I and reduce:

$$\left(\begin{array}{ccc|ccc} 0 & -1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right)$$

So $P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$. It would be a good idea to check that $PP^{-1} = P^{-1}P = I$. □

Step 5: Show that $A = PDP^{-1}$.

Solution.

$$\begin{aligned} PDP^{-1} &= \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \\ &= A \end{aligned}$$

□

Step 6: Find A^{13}. You do not need to simplify, just write out the matrix.

Solution.

$$\begin{aligned} A^{13} &= (PDP^{-1})^{13} = PD^{13}P^{-1} \\ &= \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{13} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2^{13} & -2 \\ 2^{13} & 0 & 1 \\ 0 & 2^{13} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2^{13}+2 & 0 & -2^{14}+2 \\ 2^{13}-1 & 2^{13} & 2^{13}-1 \\ 2^{13}-1 & 0 & 2^{14}-1 \end{pmatrix} \end{aligned}$$

□