1. Check to see if the vectors are linearly independent:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \xrightarrow{-4R1+R2\to R2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \xrightarrow{\frac{1}{4}R3\to R3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-R3+R2\to R2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This system has the trivial solution $c_1 = c_2 = c_3 = 0$, so the vectors are linearly independent.

Check to see if the vectors span the space: Using the same row operations as above, we can rewrite our corresponding matrix as

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{as} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since there is a pivot in every row, the vectors span the space.

Since the vectors are linearly independent and span the space, then **yes**, the vectors form a basis for $\mathbb{R}^3$.

2. Let A denote the given matrix. Solve the system $A\mathbf{x} = \mathbf{0}$ by augmenting the original matrix with the zero vector and reducing:

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 2 & -4 & 8 & 0 \end{array}\right) \xrightarrow{-2R1+R2\to R2} \left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_2, x_3 are free variables. Then $x_1 = 2x_2 - 4x_3, x_2 = x_2, x_3 = x_3$, and we write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} x_3.$$

Hence the nullspace of A is given by

$$null(A) = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \right\},$$

and the basis of the nullspace of A is

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

3. Let A denote the given matrix. Reduce A:

$$\begin{pmatrix} 1 & -2 & 4 \\ 2 & -4 & 8 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-2R1+R2\to R2} \begin{pmatrix} 1 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R2\leftrightarrow R3} \begin{pmatrix} 1 & -2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The first and third columns have pivots, hence the column space of A is the span of the first and third columns of the original matrix A; that is,

$$col(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix} \right\}.$$

The basis of the column space of A is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix} \right\}.$$

4. We need to see if the given vector is a linear combination of the other two vectors, i.e. we need to see if

$$a \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

has a solution. This is the same as checking of the following system has a solution:

$$\left(\begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array} \right) \xrightarrow{-2R1 + R2 \to R2} \left(\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 3 & 6 & 9 \end{array} \right) \xrightarrow{-3R1 + R3 \to R2} \left(\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{array} \right)$$

$$\xrightarrow{-(1/3)R2 \leftrightarrow R2, -(1/6)R3 \leftrightarrow R3} \left(\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} \right) \xrightarrow{R3 - R2 \to R3} \left(\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{-4R2 + R1 \to R1} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

Since the system has a solution, namely $a = -1, b = 2$, then **yes**, the given vector in the span.

5. Augment a matrix of the basis vectors of $\mathcal{B}$ with a matrix of the basis vectors of $\mathcal{C}$ and reduce:

$$\left(\begin{array}{cc|cc} 1 & 3 & 3 & 1 \\ 2 & 4 & 4 & 2 \end{array} \right) \xrightarrow{-2R1 + R2 \to R2} \left(\begin{array}{cc|cc} 1 & 3 & 3 & 1 \\ 0 & -2 & -2 & 0 \end{array} \right) \xrightarrow{-(1/2)R2 \to R2} \left(\begin{array}{cc|cc} 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

$$\xrightarrow{-3R2 + R1 \to R1} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

Thus $P_{\mathcal{B} \to \mathcal{C}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

6. (a) **True**. A basis for $\mathbb{R}^2$ has two vectors. This means at most two vectors in $\mathbb{R}^2$ are linearly independent, and anymore than two vectors is linearly dependent.

(b) **True**. It the rank of a 3×2 matrix has rank 2, then nullity = 2 - 2 = 0. Since we have nullity = 0, then the only solution is the trivial solution.

Another way: Since our matrix has rank 2, then it has two vectors in the column space. These vectors correspond with two pivots in our matrix. Since the number of pivots is equal to the number of columns, then there are no free variables. Since there are no free variables, then the only element in the nullspace is the trivial solution.

(c) **False**. The unit disk is given by $U = \{(x, y) : x^2 + y^2 \leq 1\}$. The unit disk is not closed under scalar multiplication since $(1, 0) \in U$, but $5(1, 0) = (5, 0) \notin U$. (There are many different scalars you can choose.)

Another way: The unit disk is not closed under addition since $(1, 0) \in U$ and $(0, 1) \in U$, but $(1, 0) + (0, 1) = (1, 1) \notin U$.