1. Set up the augmented system and reduce using row operations:

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 3 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow{R2-R1\to R2} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow{R1\leftrightarrow R2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-2R1+R2\to R2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow{R3-R2\to R3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There is one free variable, let z be free. From the first and second row we have

$$\begin{aligned} x+z=0 &\quad\Rightarrow\quad x=-z \\ y-z=1 &\quad\Rightarrow\quad y=z+1 \end{aligned}$$

Write the general solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z+1 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} z.$$

2. We need to see if there is an a, b such that

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Set up the augmented system and reduce using row operations:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{array}\right) \xrightarrow{R2-R1\to R2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 1 & 3 & 1 \end{array}\right) \xrightarrow{R3-R1\to R3} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{array}\right)$$

$$\xrightarrow{R3-2R2\to R2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R3-2R2\to R2} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

Thus $a = 4, b = -1$. Since there is a solution, then **yes**, the vector is in the span of the other two vectors.

3. We need to see if the equation

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

has the solution $c_1 = c_2 = c_3$. Set up the augmented system and reduce using row operations:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R3-R2\to R2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R4-R2\to R2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R4 \leftrightarrow R3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R2 - R3 \rightarrow R2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R1 - R3 \rightarrow R1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R1 - R2 \rightarrow R1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $c_1 = c_2 = c_3 = 0$, hence the vectors are **linearly independent**.

4. One-to-one means the kernel of the transformation, i.e. the nullspace of the matrix, has only the trivial solution (the zero vector). Find the nullspace of the given transformation matrix by augmenting the matrix with the zero vector and reducing using row operations:

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R3 \rightarrow R3} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R1 - R3 \rightarrow R3} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R2 \leftrightarrow R3} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The solution to this system is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

hence the nullspace is trivial solution. Thus the transformation is **one-to-one**.

Onto means the vectors of the transformation matrix span the space, i.e. there is a vector in every row. From the row reduction above, we see that there is not a pivot in every row, hence the transformation is **not onto**.

5.

$$A^2 = AA = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 4 \end{pmatrix}$$

Use the inverse formula for 2×2 matrices to calculate A^{-1}:

$$A^{-1} = \frac{1}{1(2) - 0(2)} \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1/2 \end{pmatrix}$$

$$det(A) = 1(2) - 0(2) = 2$$

6. Find the column space of the corresponding matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} \xrightarrow{R2 - R1 \rightarrow R2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 1 & 3 & 2 \end{pmatrix} \xrightarrow{R3 - R1 \rightarrow R3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{R2 - 2R3 \rightarrow R2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The first and second columns have pivots, so the first and second columns of the original matrix form the basis:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \right\}$$

7. Augment the matrix with the zero vector and solve for the general solution:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

There are two free variables, let x_2, x_4 denote the free variables. From the first and second row we have

$$\begin{aligned} x_1 + x_4 = 0 \quad &\Rightarrow \quad x_1 = -x_4 \\ x_3 + x_4 = 0 \quad &\Rightarrow \quad x_3 = -x_4 \end{aligned}$$

Write the general solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} x_4$$

The basis of the nullspace is therefore

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

8. (a) True. A basis of $\mathbb{R}^3$ contains three elements, which means at most three vectors can be linearly independent in $\mathbb{R}^3$.
   (b) False, can you come up with a counterexample?
   (c) False, can you come up with a counterexample?
   (d) True. A basis of $\mathbb{R}^3$ contains three elements, which means at least three vectors are needed to span $\mathbb{R}^3$.
   (e) True.

9. If $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathbf{x}_B = \begin{pmatrix} a \\ b \end{pmatrix}$, then

$$\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2.$$

Therefore

$$v = 1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

10. $\mathbf{v}$ is an eigenvector of A with eigenvalue λ if $A\mathbf{v} = \lambda\mathbf{v}$. Multiply the given matrix by the given vector and check if the resulting vector is a multiple of the given vector:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Yes, the given vector is an eigenvector of the given matrix. (Fun fact: the corresponding eigenvalue is $\lambda = 4$.)

11. See "Complex Eigenvalues" notes on Gauchospace to find another example. Find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - (-2) = \lambda^2 - 2\lambda + 3$$

Use the quadratic formula:

$$\lambda = \frac{2 \pm \sqrt{2^2 - 4(1)(3)}}{2} = 1 \pm i\sqrt{2}$$

Therefore $\lambda_1 = 1 + i\sqrt{2}, \lambda_2 = 1 - i\sqrt{2}$.

Find the eigenvector for λ_1:

$$\left(\begin{array}{cc|c} 1-\lambda_1 & -2 & 0 \\ 1 & 1-\lambda_1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1-(1+i\sqrt{2}) & -2 & 0 \\ 1 & 1-(1+i\sqrt{2}) & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -i\sqrt{2} & -2 & 0 \\ 1 & -i\sqrt{2} & 0 \end{array}\right)$$

$$\xrightarrow{-i\sqrt{2}R1 \to R1} \left(\begin{array}{cc|c} 2 & -2i\sqrt{2} & 0 \\ 1 & -i\sqrt{2} & 0 \end{array}\right) \xrightarrow{-2R2 + R1 \to R1} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -i\sqrt{2} & 0 \end{array}\right)$$

There is one free variable. The second equation tells us $x_1 - i\sqrt{2}x_2 = 0$, or that $x_1 = i\sqrt{2}x_2$. Let x_2 be the free variable. Then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} i\sqrt{2}x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} i\sqrt{2} \\ 1 \end{pmatrix} x_2,$$

thus

$$\mathbf{v}_1 = \begin{pmatrix} i\sqrt{2} \\ 1 \end{pmatrix} = i\begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Form the matrix P by $P = (Im(\mathbf{v}_1) \quad Re(\mathbf{v}_1))$:

$$P = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$$

and find its inverse:

$$P^{-1} = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$$

Find $P^{-1}AP$:

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} Re(\lambda_1) & -Im(\lambda_1) \\ Im(\lambda_1) & Re(\lambda_1) \end{pmatrix} \end{aligned}$$

Find the magnitude of λ_1:

$$|\lambda_1| = \sqrt{Re(\lambda_1)^2 + Im(\lambda_1)^2} = \sqrt{1^2 + \sqrt{2}^2} = \sqrt{3}$$

The standard form of the matrix is

$$A = \begin{pmatrix} |\lambda_1| & 0 \\ 0 & |\lambda_1| \end{pmatrix} \begin{pmatrix} Re(\lambda_1)/|\lambda_1| & -Im(\lambda_1)/|\lambda_1| \\ Im(\lambda_1)/|\lambda_1| & Re(\lambda_1)/|\lambda_1| \end{pmatrix},$$

or, in our case:

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & -\sqrt{2}/\sqrt{3} \\ \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The left matrix is the expansion, the right matrix is the rotation.

12.

$$||\mathbf{u}|| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1(1) + (-1)(1) + 2(1) = 2$$

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-1)^2 + (-1-1)^2 + (2-1)^2} = \sqrt{5}$$