Math 4B Notes

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Classification of Differential Equations

The **order** of a differential equation is the order of the highest derivative that appears in the equation. For example, the differential equation

$$\frac{d^3y}{dx^3} + x^2y\frac{d^5y}{dx^5} - \sin^3(xy)\frac{dy}{dx} = \cot(x)$$

has order 5.

An ordinary differential equation is said to be **linear** if it can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = g(x)$$

where $g(x)$ and each $a_i(x)$ are functions of x and not y. For example, the differential equation

$$y''' + xy^2y' = x^4$$

is not linear because of the term xy^2y'.

Separable Equations

A separable equation is a first order linear differential equation that can be written as

$$\frac{dy}{dx} = f(x)g(y).$$

Use the following steps to solve the equation:

1. Write a function of x on one side of the equation and a function of y on the other side of the equation, e.g. the above equation would look like $\frac{dy}{g(y)} = f(x)dx$.
2. Integrate both sides.
3. Solve for y if possible and C if given an initial value.

First Order Linear Equations

A first order linear differential equation is of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Use the following steps to solve the equation:

1. Write the equation in **standard form**: $\dfrac{dy}{dx} + P(x)y = f(x)$.
2. Find the integrating factor $\mu(x) = e^{\int P(x)dx}$.
3. Multiply the standard equation in Step 1 by $\mu(x)$. Your equation can now be written in the form $(\mu(x)y)' = \mu(x)f(x)$.
4. Integrate: $\mu(x)y = \int \mu(x)f(x)dx$.
5. Solve for y and C if given an initial value: $y = \dfrac{\int \mu(x)f(x)dx}{\mu(x)}$.

Friendly note: While you *could* just memorize the formula in Step 5, you shouldn't. It is better to follow the steps listed above and understand what you are solving rather than just plug in your functions into a formula.

Exact Equations and Integrating Factors

An equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \qquad \text{or} \qquad M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is said to be an **exact equation** if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Our goal is to find a function $f(x,y)$ such that:

$$\frac{\partial f}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N(x,y).$$

Use the following steps to solve an **exact** equation:

1. Verify that $M_y = N_x$.
2. Since we want $\frac{\partial f}{\partial x} = M(x,y)$, integrate with respect to x:

$$f(x,y) = \int M(x,y)dx + g(y).$$

3. Since we want $\frac{\partial f}{\partial y} = N(x,y)$, differentiate what you found in Step (2) with respect to y:

$$\frac{\partial}{\partial y}\int M(x,y)dx + g'(y) = N(x,y).$$

4. Solve for $g'(y)$:

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx$$

   and integrate with respect to y on both sides.

5. Plug your solution for $g(y)$ into your equation for step (2). The solution is $f(x,y) = C$.

The Existence and Uniqueness Theorem (for first order equations)

Theorem (Existence and Uniqueness Theorem for First Order Linear Equations)**.** *If the functions P and f are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation $y' + P(t)y = f(t)$ for t in I, and that also satisfies the initial condition $y(t_0) = y_0$, where y_0 is an arbitrary prescribed value.*

What this means if that P and f are continuous on its domain, and your initial value t_0 is in some interval in that domain, then there exists a unique solution to that differential equation in that interval.

For example, consider the differential equation

$$y' + \frac{1}{x-1}y = \frac{1}{\sqrt{x+4}}.$$

$P(x) = \frac{1}{x-1}$ and $f(x) = \frac{1}{\sqrt{x+4}}$. The domain of $P(x)$ is everything except $x = 1$, which we can write as $(-\infty, 1) \cup (1, \infty)$. The domain of $f(x)$ is all numbers greater than -4, or $(-4, \infty)$. Taking the intersection of these domains gives us $(-4, 1) \cup (1, \infty)$. P and f are continuous on this domain. If we were given the initial value of $y(5) = y_0$, the largest interval that guarantees a solution is the interval $(1, \infty)$ since 5 is in that interval. If we were given the initial value of $y(-\pi) = y_0$, the largest interval that guarantees a solution is the interval $(-4, 1)$ since $-\pi$ is in that interval.

Homogeneous Linear Equations With Constant Coefficients

Consider the special case of the second order equation $ay'' + by + cy = 0$. We want to find a solution of the form $y = e^{mx}$, and by substitution this into the equation we get

$$a(e^{mx})'' + b(e^{mx})' + c(e^{mx}) = 0 \quad \Rightarrow \quad am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \Rightarrow \quad e^{mx}(am^2 + bm + c) = 0.$$

Since e^{mx} never equals zero, we have

$$am^2 + bm + c = 0$$

and we can now solve for m. This equation above is called the **auxiliary** or **characteristic** equation of our differential equation.

You don't need to use this derivation all the time to get the characteristic/auxiliary equation. Just know that $y^{(n)}$ corresponds with m^n; thus an equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

has the characteristic/auxiliary equation

$$a_n m^n + a_{n-1} m^{n-1} + ... + a_1 m + a_0 = 0.$$

Use the following steps to solve a homogeneous linear equation with constant coefficients:

1. Write the characteristic/auxiliary equation for the differential equation.
2. Solve for your zeros m.
3. Write the general solution according to the cases below :
   (a) If m is real and has multiplicity 1 (distinct from all the other values), then the solution is of the form ce^{mx}.
   (b) If m is real and has multiplicity k, then the solution is of the form $c_1 e^{mx} + c_2 x e^{mx} + ... + c_k x^{k-1} e^{mx}$.
   (c) If m is complex (i.e., $m = \alpha + i\beta$), then the general solution is of the form $e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)$.

   Note: you may have one or more of the cases depending on your polynomial. If so, add them together to form your entire general solution.
4. Solve for your constants if given initial values.

The Wronskian

The Wronskian of a set of functions $f_1, f_2, ..., f_n$ is the determinant

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix}.$$

The set of functions $f_1, f_2, ..., f_n$ are said to form a fundamental set of solutions to a differential equation if and only if they are solutions to that differential equation on some interval and $W(f_1, f_2, ..., f_n) \neq 0$ for every x in the solution interval.

Reduction of Order

Consider the homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

and suppose that we know y_1 is a solution to the equation above. Using the method of reduction of order we can find a second solution y_2 to the differential equation, where $y_2(x) = v(x)y_1(x)$.

Use the following steps to solve such an equation using reduction of order:

1. Write the differential equation in standard form: $y'' + P(t)y' + Q(t)y = 0$.
2. Let $y_2 = vy_1$. Find y_2' and y_2'' using the product rule, and plug these into the given differential equation. You should get a second order equation in terms of v:

   $$y_1v'' + (2y_1' + P(t)y_1)v' = 0.$$

   (Remember to use the fact that $y'' + P(t)y' + Q(t)y = 0$!)
3. Let $w = v'$, then $w = v''$. Plug these into the equation above:

   $$y_1w' + (2y_1' + P(t)y_1)w = 0.$$

   This is a first order linear equation, use the appropriate method to solve for w.
4. Integrate w to find v: $v = \int wdx$.
5. The second solution to the equation is $y_2 = vy_1$, and the general solution to the differential equation is $y = c_1y_1 + c_2y_2$.

Undetermined Coefficients

Consider the nonhomogeneous equation $y''+P(x)y+Q(x)y = g(x)$ where $P(x), Q(x)$ are constants. If $g(x)$ is of the type

$$p(x) = a_nx^n + ... + a_1x + a_0, \quad p(x)e^{\alpha x}, \quad p(x)e^{\alpha x}\sin(\beta x), \quad p(x)e^{\alpha x}\cos(\beta x),$$

we can use a method called the "method of undetermined coefficients". There are actually two methods using undetermined coefficients, one is a substitution approach and the other is called the annihilator approach. In these notes I will only be discussing the substitution approach.

Here are some example of the form we use for y_p:

$g(x)$	Form of y_p
x^3+x^2+x+2	Ax^3+Bx^2+Cx+E
e^{5x}	Ae^{5x}
$\sin(4x)$	$A\sin(4x)+B\cos(4x)$
x^2e^x	$(Ax^2+Bx+C)e^x$
$5x\sin 2x$	$(Ax+B)\sin 2x+(Cx+E)\cos 2x$
$x^2e^x\sin x$	$(Ax^2+Bx+C)e^x\sin x+(Ex^2+Fx+G)e^x\cos x$

We need to be careful with how we pick y_p, however. If $g(x)$ does not have any functions contained in the solution to the homogeneous equation, then y_p is found similar to the equations above. However, if $g(x)$ does contain functions that are in the general solution, then we must multiply our y_p by x^n where n is the smallest integer that eliminates duplication.

For example, consider the equation $y''-2y'+1=e^x$. The general solution to the homogeneous equation $y''-2y'+1=0$ is $y_c=c_1e^x+c_2xe^x$. Notice, however, that $g(x)=e^x$ which is already in our general solution, so we can't use $y_p=e^x$. If we multiply by x, xe^x is still in our general solution, so we need to multiply by x again. Thus the form we would want to use for $y_p=x^2e^x$.

Use the following steps to solve nonhomogeneous linear equations using the method of undetermined coefficients:

1. Put into standard form $y''+P(x)y'+Q(x)=g(x)$.
2. Solve the associated homogeneous equation to get the complementary or general solution y_c (or sometimes denoted y_h).
3. Find the form of y_p using the following cases:
   (i) $g(x)$ contains no function of y_c
   (ii) $g(x)$ contains a function of y_c
4. Substitute y_p into your equation and solve for the coefficients.
5. The solution to the differential equation is $y=y_c+y_p$.

Variation of Parameters

Consider the nonhomogeneous equation $y''+P(x)y+Q(x)y=g(x)$ where $P(x),Q(x)$ are constants. We can use a method called Variation of Parameters to solve this equation.

Use the following steps to solve nonhomogeneous linear second order equations using the method of variation of parameters:

1. Put into standard form $y''+P(x)y'+Q(x)=g(x)$.

2. Solve the associated homogeneous equation to get the complementary or general solution $y_c = c_1y_1 + c_2y_2$ (or sometimes denoted y_h).*
3. Find W, W_1, and W_2 where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

4. Find u_1, u_2 where

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

5. The particular solution is $y_p = u_1y_1 + u_2y_2$.
6. The solution to the differential equation is $y = y_c + y_p$.

*Sometimes the homogeneous solution to $y'' + P(x)y' + Q(x)y = 0$ may be given if P, Q are functions of x rather than constants.

Note: there is also a formula that you may memorize for Variation of Parameters.

Model of Spring System

A spring is described by the equation

$$mu''(t) + \gamma u'(t) + ku(t) = f(t)$$

where m is the mass attached to the end of the spring, γ is the damping or friction constant, k is the spring constant found from Hooke's Law ($F = k\Delta x$) and $f(t)$ is the external force.

If $\gamma = 0$, the system is **undamped**. If $\gamma \neq 0$, and the characteristic equation to the homogeneous equation has

- two real distinct roots, the system is **overdamped**.
- one real repeated root, the system is **critically damped**.
- two complex roots, the system is **underdamped**.

Homogeneous Linear Systems

A homogeneous linear first order system will be of the form $\mathbf{x}' = A\mathbf{x}$. For these notes we will assume A is a 2×2 matrix.

We have the following cases:

(i) A has distinct eigenvalues

(ii) A has repeated eigenvalues

(iii) A has complex eigenvalues

Use the following steps to solve:

1. Find the eigenvalues
2. Find the eigenvectors
3. Write the general solution based on the following cases above:

   (i) If λ_1, λ_2 are real, distinct eigenvalues with eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, the general solution is of the form
   $$\mathbf{x} = c_1\mathbf{v_1}e^{\lambda_1 t} + c_2\mathbf{v_2}e^{\lambda_2 t}.$$

   (ii) If λ is a real, repeated eigenvalue with eigenvector $\mathbf{v}_1$, the general solution is of the form
   $$\mathbf{x} = c_1\mathbf{v_1}e^{\lambda t} + c_2e^{\lambda t}(\mathbf{v_1}t + \mathbf{v_2}).$$
   where $\mathbf{v}_2$ is the **generalized eigenvector** from the equation $(A - I\lambda)\mathbf{v_2} = \mathbf{v_1}$.

   (iii) If λ_1, λ_2 are complex eigenvalues (they should be complex conjugates) with eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ (they should be complex conjugates), the general solution is of the form
   $$\mathbf{x} = c_1\mathrm{Re}(\mathbf{v_1}e^{\lambda_1 t}) + c_2\mathrm{Im}(\mathbf{v_1}e^{\lambda_1 t}).$$
   Note: $\mathbf{v_2}e^{\lambda_2 t}$ will have similar real and imaginary parts which is why we only care about $\mathbf{v_1}e^{\lambda_1 t}$.

Note: It is possible that we can have repeated eigenvalues which return multiple eigenvectors in larger systems. For example, the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\mathbf{x}$$

has an eigenvalue $\lambda = -1$ with multiplicity 2 with two linearly independent eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Its general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}.$$

It is when the number of eigenvectors is less than the multiplicity of the eigenvalue that we must find more eigenvectors using the approach in Case (ii).

For a 3×3 system with eigenvalue λ of multiplicity 3 but only one eigenvector, the general solution is

$$\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda t} + c_2(\mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}) + c_3 \left(\mathbf{v}_1 \frac{t^2}{2} e^{\lambda t} + \mathbf{v}_2 t e^{\lambda t} + \mathbf{v}_3 e^{\lambda t} \right)$$

where $(A - \lambda I)\mathbf{v}_1 = \mathbf{0}, (A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1, (A - \lambda I)\mathbf{v}_3 = \mathbf{v}_2$.

Fundamental Matrices

A **fundamental matrix** is a matrix of your linearly independent solutions. For example, consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}.$$

This system has the solution

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t},$$

therefore the fundamental matrix of this system is

$$\psi = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}.$$

Undetermined Coefficients for Nonhomogeneous Linear Systems

A nonhomogeneous linear system is of the form $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$ where $\mathbf{g}(t)$ is a vector. If $\mathbf{g}(t)$ is a polynomial, exponential function, or sine or cosine function, then we can use the method of undetermined coefficients to make an educated guess about the particular solution $\mathbf{x}_p$.

Use the following steps to solve a nonhomgeneous linear system using the method of undetermined coefficients:

1. Write the equation in standard form: $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$.

2. Find the homogeneous solution $\mathbf{x}_h$ to the system $\mathbf{x}' = A\mathbf{x}$.

3. Guess $\mathbf{x}_p$ using the following cases:

   (i) $g(t)$ contains no function in common with $\mathbf{x}_h$

   (ii) $g(t)$ contains a function in common with $\mathbf{x}_h$

4. The solution is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$.

Guessing for $\mathbf{x}_p$ is similar to undetermined coefficients for a nonhomogeneous linear equation with constant coefficients, however our coefficients are now vectors. See the table below for examples.

$\mathbf{g}(t)$	Form of $\mathbf{x}_p$
$\begin{pmatrix} 1 \\ 2 \end{pmatrix} t^2 + \begin{pmatrix} 10 \\ -3 \end{pmatrix}$	$\mathbf{a}t^2 + \mathbf{b}t + \mathbf{c} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t^2 + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$
$\begin{pmatrix} 8 \\ 0 \end{pmatrix} e^{5t}$	$\mathbf{a}e^{5t} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{5t}$
$\begin{pmatrix} 5 \\ 1 \end{pmatrix} \sin 4t + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 4t$	$\mathbf{a}\sin(4t) + \mathbf{b}\cos(4t) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sin 4t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cos 4t$

Similar to nonhomogeneous linear equations with constant coefficients, we need to be careful if $\mathbf{g}(\mathrm{t})$ contains functions in common with the homogeneous solution $\mathbf{x}_h$. However, the rules we used for constant coefficient equations will not be quite the same for linear systems as seen in the following two examples.

Example 1. Find the form of $\mathbf{x}_p$ for the equation $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 8 \\ -3 \end{pmatrix}$ given that the homogeneous solution is $\mathbf{x}_h = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Solution. Since $\mathbf{g}(t) = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$, we would guess $\mathbf{x}_p = \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. However, notice that a vector of constants is already in the solution $\mathbf{x}_h$ (it's the term $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$). Therefore we must guess

$$\mathbf{x}_p = \mathbf{a}t + \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

□

Example 2. Find the form of $\mathbf{x}_p$ for the equation $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 8 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$ given that the homogeneous solution is $\mathbf{x}_h = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Solution. Since $\mathbf{g}(t) = \begin{pmatrix} 8 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$, we would guess $\mathbf{x}_p = \mathbf{a} + \mathbf{b}e^{3t} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{3t}$. However, notice that a vector of constants is already in the solution $\mathbf{x}_h$ (it's the term $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$) and a vector multiplied by e^{3t} is already in the solution $\mathbf{x}_h$ (it's the term $c_2e^{3t}\begin{pmatrix}-1\\1\end{pmatrix}$). Therefore we must guess

$$\mathbf{x}_p = \mathbf{a}t + \mathbf{b} + \mathbf{c}te^{3t} + \mathbf{d}e^{3t} = \begin{pmatrix}a_1\\a_2\end{pmatrix}t + \begin{pmatrix}b_1\\b_2\end{pmatrix} + \begin{pmatrix}c_1\\c_2\end{pmatrix}te^{3t} + \begin{pmatrix}d_1\\d_2\end{pmatrix}e^{3t}.$$

□

Stability and Classification of Homogeneous Linear Systems

We will be considering a homogeneous linear system of the form $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix. You will need to solve for your eigenvalues of your matrix A. You should also write out the general solution and sketch a phase plane diagram. If your phase plane diagram "flows inward" and the the limit as $t \to \infty$ of your solution converges, the solution is **stable**. If the phase plane diagram "flows outward" and the limit as $t \to \infty$ of your solution diverges, the solution is **unstable**. Below are some general cases of classifying the behavior of a homogeneous linear system:

(i) Real and distinct eigenvalues

- If λ_1, λ_2 are the same sign, we have a **node**. If they are both positive, then the solution is unstable (sketch it). If they are both negative, the solution is stable (sketch it).
- If λ_1, λ_2 are opposite signs, we have a **saddle**. Saddles are unstable (sketch it).

(ii) Real and repeated eigenvalues

- If λ is a repeated eigenvalue with two eigenvectors we have a **star**.
- If λ is a repeated eigenvalue with one eigenvector we have a **degenerate node**. If λ is negative, the solution is stable. If λ is positive, the solution is unstable.

(iii) Complex eigenvalues

- If the real part is equal to zero, then we have a **center**.
- If the real part is nonzero, then we have a **spiral**. If the real part is negative, then the solution is stable (sketch it). If the real part is positive, the solution is unstable (sketch it).

Linearization and Local Stability

Consider the system

$$\frac{dx}{dt} = P(x,y)$$
$$\frac{dy}{dt} = Q(x,y)$$

Equilibrium (or fixed) points are when $P(x,y) = 0$ and $Q(x,y) = 0$. The Jacobian matrix J is defined to be

$$J = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}.$$

The Jacobian matrix will help us identify the stability at each equilibrium point.

Use the following steps to classify behavior of equilibrium points of a nonlinear system:

1. Find the equilibrium points (x_0, y_0) by setting $P(x,y) = 0$ and $Q(x,y) = 0$.
2. Find J and evaluate $J(x_0, y_0)$ at each equilibrium point.
3. Find the eigenvalues of $J(x_0, y_0)$ at each equilibrium point.
4. Use the eigenvalues to classify the behavior and stability at that equilibrium point (see previous section on stability and classification).

If you are to sketch the phase plane, plot the equilibrium points. Then find the corresponding eigenvectors to the eigenvalues and plot the behavior on the graph.