Practice Problems III

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Answers

This page contains answers only. Detailed solutions are on the following pages.

1. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}$

2. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right] + c_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix} e^{2t} \right]$

3. $\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{6t} + c_2 e^{4t} \begin{pmatrix} \cos 2t \\ 0 \\ -2\sin 2t \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} \sin 2t \\ 0 \\ 2\cos 2t \end{pmatrix}$

4. (a) $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2}$
   (b) Saddle, unstable
   (c) $\mathbf{x} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2}$

5. (a) $\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + c_2 e^{4t} \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right)$
   (b) Degenerate node, unstable
   (c) $\mathbf{x} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} e^{4t} + \begin{pmatrix} 26 \\ 13 \end{pmatrix} te^{4t}$

6. (a) $\mathbf{x} = c_1 e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix}$
   (b) Spiral, unstable
   (c) $\mathbf{x} = e^{5t} \begin{pmatrix} -2\cos 2t - 5\sin 2t \\ 8\cos 2t - 9\sin 2t \end{pmatrix}$

7. $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p = c_1 e^{3t} \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^t$

8. (a) $(0, -2), (2, 2), (-1, -1)$
   (b) $J = \begin{pmatrix} -2x & 1 \\ 2x - y & -x \end{pmatrix}$
   (c) $(0, -2)$ unstable saddle
   $(2, 2)$ stable node
   $(-1, -1)$ unstable spiral

Detailed Solutions

1. Solve $\mathbf{x}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{x}$.

Solution. We begin by finding the eigenvalues:

$$\begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & -2 \\ 2 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 1-\lambda \\ 2 & -2 \end{vmatrix}$$

$$\begin{aligned}
&= (1-\lambda)[(1-\lambda)(1-\lambda)-4] + 2[-2(1-\lambda)+4] + 2[4-2(1-\lambda)] \\
&= (1-\lambda)(\lambda^2-2\lambda-3) + 2(2\lambda+2) + 2(2\lambda+2) \\
&= (1-\lambda)(\lambda-3)(\lambda+1) + 4(\lambda+1) + 4(\lambda+1) \\
&= (\lambda+1)\,[(1-\lambda)(\lambda-3)+4+4] \\
&= (\lambda+1)\left[-\lambda^2+4\lambda+5\right] \\
&= -(\lambda+1)(\lambda+1)(\lambda-5)
\end{aligned}$$

Therefore the eigenvalues are $\lambda = -1$ mult. 2, $\lambda = 5$.

Now we find the eigenvectors. Start with $\lambda = -1$:

$$\left(\begin{array}{ccc|c} 1-(-1) & -2 & 2 & 0 \\ -2 & 1-(-1) & -2 & 0 \\ 2 & -2 & 1-(-1) & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have two rows of zeros, hence we have two free variables. The first row tells us that $x_1 = x_2 - x_3$. We can write the general solution of the system as

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} x_3$$

Choose $x_2 = 1, x_3 = 0$, and then $x_2 = 0, x_3 = 1$ to get the eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Notice that we have 2 eigenvectors which matches the multiplicity of the eigenvalue $\lambda = -1$. This means two things: we do not need to go find any more eigenvectors, AND we do not need to have a te^{-t} as part of our solution because we found two independent eigenvectors (i.e. we do not need to "bump up" the solution). Now we find the eigenvector for when $\lambda = 5$:

$$\left(\begin{array}{ccc|c} 1-5 & -2 & 2 & 0 \\ -2 & 1-5 & -2 & 0 \\ 2 & -2 & 1-5 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(We skipped a couple of steps.) The first row tells us that $x_1 = x_2 + 2x_3$, and the second row tells us that $x_2 = -x_3$. If we choose $x_3 = 1$, then $x_2 = -1$ and $x_1 = 1$. Therefore the third eigenvector is

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}.$$

□

2. Solve $\mathbf{x}' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$.

Solution. We begin by finding the eigenvalues. Because our matrix is an upper triangular matrix, the eigenvalues are the diagonal entries, hence $\lambda = 2$ mult. 3. We now find the eigenvectors for $\lambda = 2$:

$$\left(\begin{array}{ccc|c} 2-2 & 1 & 6 & 0 \\ 0 & 2-2 & 5 & 0 \\ 0 & 0 & 2-2 & 0 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The second row tells us that $x_3 = 0$, and the first row tells us that $x_2 = -6x_3$. But this implies that $x_2 = 0$. x_1 is our free variable, so we can choose it to be 1, and we get the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We only found one eigenvector, which is less than the multiplicity of our eigenvalue. This means two things: we must find more eigenvectors, and we must have a t^2e^{2t} and te^{2t} in our solution (i.e. we must "bump up" our solution). We find the next eigenvector by solving the equation $(A - I\lambda)\mathbf{v}_2 = \mathbf{v}_1$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The second row tells us $x_3 = 0$, and the first row tells us that $x_2 = -6x_3 + 1$, which means that $x_2 = 1$. x_1 is free, so if we choose it to be 0 then the second eigenvector is

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We need to find another eigenvector, so we solve the equation $(A - I\lambda)\mathbf{v}_3 = \mathbf{v}_2$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The second row tells us that $x_3 = \frac{1}{5}$, the first row tells us that $x_2 = -6x_3 = -\frac{6}{5}$. x_1 is free, so if we choose it to be 0 then the third eigenvector is

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix}.$$

Therefore the solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right] + c_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix} e^{2t} \right].$$

□

3. Solve $\mathbf{x}' = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 6 & 0 \\ -4 & 0 & 4 \end{pmatrix} \mathbf{x}$.

Solution. We begin by finding the eigenvalues:

$$\begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 6-\lambda & 0 \\ -4 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 6-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} - 0 + 1\begin{vmatrix} 0 & 6-\lambda \\ -4 & 0 \end{vmatrix}$$
$$= (4-\lambda)(6-\lambda)(4-\lambda) + 4(6-\lambda)$$
$$= (6-\lambda)[(4-\lambda)(4-\lambda)+4]$$
$$= (6-\lambda)(\lambda^2 - 8\lambda + 20)$$

We use the quadratic formula to find the eigenvalues of $\lambda^2 - 8\lambda + 20$:

$$\lambda = \frac{8 \pm \sqrt{64 - 4(1)(20)}}{2} = \frac{8 \pm \sqrt{-16}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$$

Therefore the eigenvalues are $\lambda_1 = 6, \lambda_2 = 4+2i, \lambda_3 = 4-2i$. When $\lambda_1 = 6$:

$$\left(\begin{array}{ccc|c} 4-6 & 0 & 1 & 0 \\ 0 & 6-6 & 0 & 0 \\ -4 & 0 & 4-6 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

The last row tells us that $x_3 = 0$, the first row tells us that $2x_1 = x_3$ which implies that $x_1 = 0$. x_2 is free, so if we choose it to be 1 then we have the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

When $\lambda_2 = 4 + 2i$:

$$\left(\begin{array}{ccc|c} 4-(4+2i) & 0 & 1 & 0 \\ 0 & 6-(4+2i) & 0 & 0 \\ -4 & 0 & 4-(4+2i) & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} -2i & 0 & 1 & 0 \\ 0 & 2-2i & 0 & 0 \\ -4 & 0 & -2i & 0 \end{array}\right)$$

The second row tells us that $(2-2i)x_2 = 0$, or that $x_2 = 0$. The first row tells us that $2ix_1 = x_3$. If we choose $x_1 = 1$, then we have the eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2i \end{pmatrix}.$$

This implies that the third eigenvector is

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -2i \end{pmatrix}.$$

Therefore the solution to the system is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2i \end{pmatrix} e^{(4+2i)t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -2i \end{pmatrix} e^{(4-2i)t},$$

however we want to write our solution as a real solution. Therefore we must expand $\mathbf{v}_2 e^{\lambda_2 t}$ (you can also do the same for $\mathbf{v}_3 e^{\lambda_3 t}$, but it is enough to just do one expansion):

$$\begin{pmatrix} 1 \\ 0 \\ 2i \end{pmatrix} e^{4t} e^{i2t} = \begin{pmatrix} 1 \\ 0 \\ 2i \end{pmatrix} e^{4t}(\cos 2t + i \sin 2t) = e^{4t} \begin{pmatrix} \cos 2t + i \sin 2t \\ 0 \\ 2i \cos 2t - 2 \sin 2t \end{pmatrix}$$

$$= e^{4t} \begin{pmatrix} \cos 2t \\ 0 \\ -2 \sin 2t \end{pmatrix} + i e^{4t} \begin{pmatrix} \sin 2t \\ 0 \\ 2 \cos 2t \end{pmatrix}.$$

Therefore the general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{6t} + c_2 e^{4t} \begin{pmatrix} \cos 2t \\ 0 \\ -2 \sin 2t \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} \sin 2t \\ 0 \\ 2 \cos 2t \end{pmatrix}.$$

□

4. Consider the system $\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}$.

(a) Find the general solution of the system.

Solution. We first find the eigenvalues. Our matrix is a lower triangular matrix, therefore the eigenvalues are the diagonal entries: $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$. You can also calculate these the usual way:

$$\begin{vmatrix} \frac{1}{2} - \lambda & 0 \\ 1 & -\frac{1}{2} - \lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda\right)\left(-\frac{1}{2} - \lambda\right) = 0 \quad \Rightarrow \quad \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}.$$

Now use the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvectors. When $\lambda_1 = \frac{1}{2}$:

$$\left(\begin{array}{cc|c} \frac{1}{2} - \frac{1}{2} & 0 & 0 \\ 1 & -\frac{1}{2} - \frac{1}{2} & 0 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right)$$

The second row tells us that $x_1 = x_2$, x_2 is free. Therefore the first eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

When $\lambda_2 = -\frac{1}{2}$:

$$\left(\begin{array}{cc|c} \frac{1}{2} + \frac{1}{2} & 0 & 0 \\ 1 & -\frac{1}{2} + \frac{1}{2} & 0 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right)$$

The first row tells us that $x_1 = 0$, x_2 is free. Therefore the second eigenvector is $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Since λ_1, λ_2 are real and distinct, then the general solution is

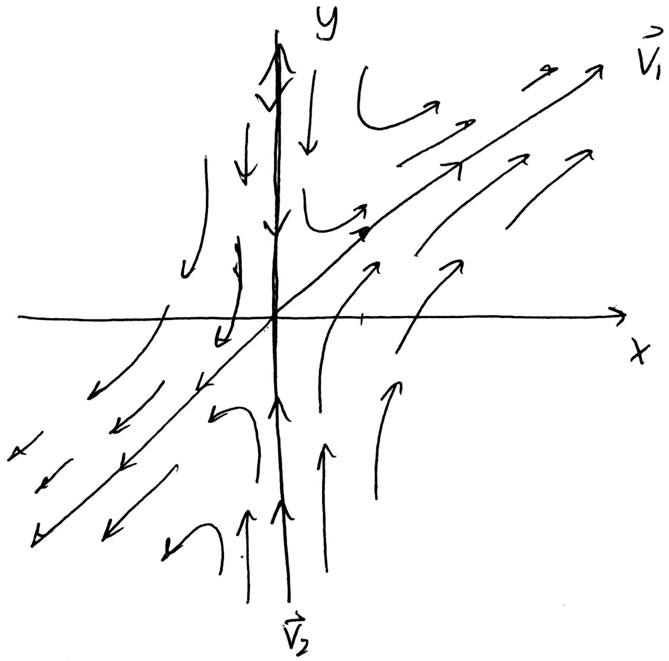
$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2}.$$

□

(b) Sketch a phase plane portrait and classify the system's geometric character and stability behavior.

Proof. Since the eigenvalues are real, distinct, and opposite sign, we have an unstable saddle. To sketch, plot both of the eigenvalues $\mathbf{v}_1$ and $\mathbf{v}_2$. Since $\lambda_1 > 0$, draw arrows pointing outward from the origin along $\mathbf{v}_1$. This is the vector that will dominate (all the solutions will want to follow this vector). Since $\lambda_2 < 0$, draw arrows pointing towards the origin along $\mathbf{v}_2$. Then fill in with arrows (see sketch below).

□



(c) Solve the given initial value problem: $\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Solution. When $t = 0$, $\mathbf{x} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$:

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which yields the system

$$\begin{aligned} 3 &= c_1 \\ 5 &= c_1 + c_2 \end{aligned}$$

The solution to this system is $c_1 = 3, c_2 = 2$. Therefore the solution is

$$\mathbf{x} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2}.$$

□

5. Consider the system $\mathbf{x}' = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \mathbf{x}$.

(a) Find the general solution of the system.

Solution. We first find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda)+4 = 12-8\lambda+\lambda^2+4 = \lambda^2-8\lambda+16 = (\lambda-4)^2 = 0 \quad \Rightarrow \quad \lambda = 4 \text{ mult. } 2$$

Now use the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvectors. When $\lambda = 4$:

$$\left(\begin{array}{cc|c} 2-4 & 4 & 0 \\ -1 & 6-4 & 0 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right)$$

The second equation tells us that $x_1 = 2x_2$, x_2 is free. Therefore we only get one eigenvector, and it is $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Since we only have one eigenvector, we will have to "bump up" our solution (i.e. there will be a te^{4t} term in our solution). We need to find a second eigenvector, so we now solve the equation $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$:

$$\left(\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right)$$

The second equation tells us that $x_1 = 2x_2 - 1$, x_2 is free. If we let $x_2 = 0$ then the second eigenvector is $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Since $\lambda = 4$ is a real repeated root and we only found one eigenvector in the beginning, then the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + c_2 \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{4t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{4t} \right).$$

□

(b) Sketch a phase plane portrait and classify the system's geometric character and stability behavior.

Proof. Since the eigenvalue is real, repeated, and positive, we have an unstable degenerate node. To sketch, plot $\mathbf{v}_1$. Since $\lambda > 0$, draw arrows pointing outward from the origin along $\mathbf{v}_1$. To figure out what direction our degenerate node is going, we will find the tangent line along a point. Consider the the point $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Plugging this into the equation yields

$$\mathbf{x}' = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

This vector points down towards the fourth quadrant, so this is where the trajectory must be moving as well. Then fill in with arrows (see sketch below).

□

(c) Solve the given initial value problem: $\mathbf{x}' = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

Solution. When $t = 0$, $\mathbf{x} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$:

$$\begin{pmatrix} -1 \\ 6 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

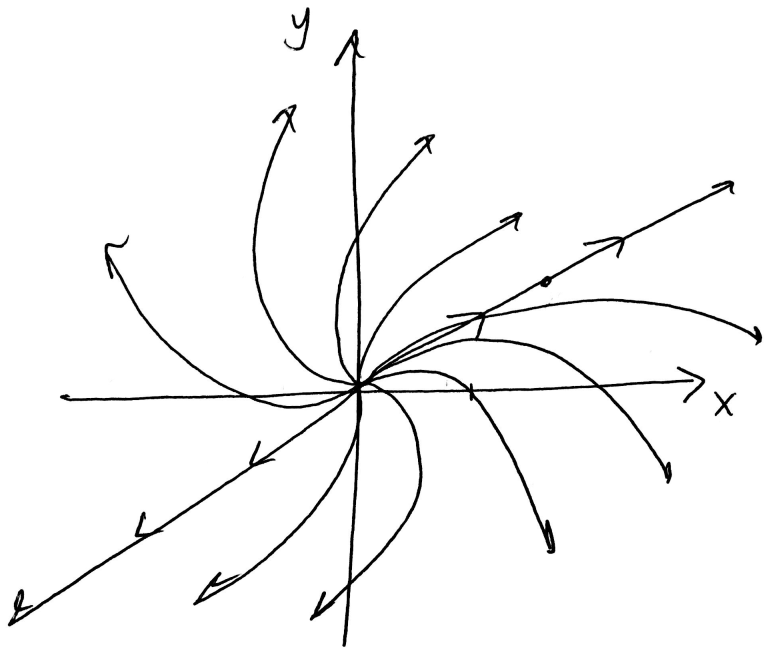
which yields the system

$$\begin{aligned} -1 &= 2c_1 - c_2 \\ 6 &= c_1 \end{aligned}$$

The solution to this system is $c_1 = 6, c_2 = 13$. Therefore the solution is

$$\begin{aligned} \mathbf{x} &= 6 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + 13 \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{4t} \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{4t} \right) \\ &= 6 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + 13 \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{4t} + 13 \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{4t} \\ &= \begin{pmatrix} -1 \\ 6 \end{pmatrix} e^{4t} + \begin{pmatrix} 26 \\ 13 \end{pmatrix} te^{4t}. \end{aligned}$$

□



6. Consider the system $\mathbf{x}' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \mathbf{x}$.

(a) Find the general solution of the system.

Solution. We first find the eigenvalues:

$$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = (6-\lambda)(4-\lambda)+5 = 24-10\lambda+\lambda^2+5 = \lambda^2-10\lambda+29 = 0$$

Use the quadratic formula to solve for λ:

$$\lambda = \frac{10 \pm \sqrt{100-4(1)(29)}}{2} = \frac{10 \pm \sqrt{-16}}{2} = \frac{10 \pm 4i}{2} = 5 \pm 2i$$

Therefore $\lambda_1 = 5+2i, \lambda_2 = 5-2i$.
Now use the equation $(A-\lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvectors. When $\lambda = 5+2i$:

$$\left(\begin{array}{cc|c} 6-(5+2i) & -1 & 0 \\ 5 & 4-(5+2i) & 0 \end{array} \right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array} \right)$$

The first row says that $(1-2i)x_1 = x_2$. If we choose $x_1 = 1$, then the first eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$. The second eigenvector will then be $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}$. The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t} + c_2 \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(5-2i)t},$$

however, we wish to write our answer as a real solution. Here we will expand both $\mathbf{v}_1 e^{\lambda_1 t}$ and

$\mathbf{v}_2 e^{\lambda_2 t}$ and see why it is enough to only expand $\mathbf{v}_1 e^{\lambda_1 t}$ to get our solution.

$$\begin{aligned}
\begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t} &= \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{5t} e^{i2t} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{5t}(\cos 2t + i \sin 2t) \\
&= e^{5t} \begin{pmatrix} \cos 2t + i \sin 2t \\ \cos 2t + i \sin 2t - 2i \cos 2t + 2 \sin 2t \end{pmatrix} \\
&= e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} + ie^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix} \\
\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(5-2i)t} &= \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{5t} e^{-i2t} = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{5t}(\cos(-2t) + i \sin(-2t)) = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{5t}(\cos 2t - i \sin 2t) \\
&= e^{5t} \begin{pmatrix} \cos 2t - i \sin 2t \\ \cos 2t - \sin 2t + 2i \cos 2t + 2 \sin 2t \end{pmatrix} \\
&= e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} - ie^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix}
\end{aligned}$$

Notice that the expansions of $\mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{v}_2 e^{\lambda_2 t}$ only differ by a sign! If you know the real and imaginary part of one of the expansions, you already know the other. This is why we have only done the expansion of $\mathbf{v}_1 e^{\lambda_1 t}$ in section. We plug these expansions into our general solution:

$$\begin{aligned}
\mathbf{x} &= c_1 \left(e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} + ie^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix} \right) + c_2 \left(e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} - ie^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix} \right) \\
&= (c_1 + c_2) e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} + i(c_1 - c_2) e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix}
\end{aligned}$$

Relabel the constants to get the general solution

$$\mathbf{x} = c_1 e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix},$$

which is just the real and imaginary parts of $\mathbf{v}_1 e^{\lambda_1 t}$. □

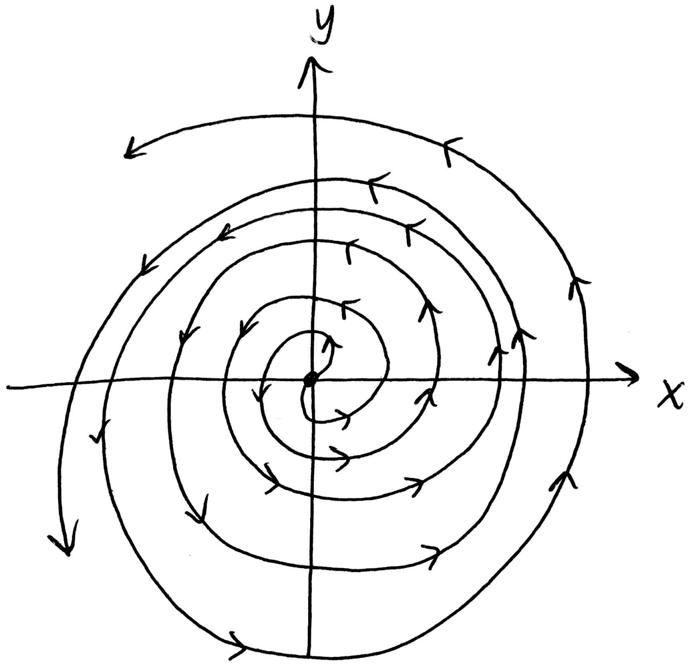
(b) Sketch a phase plane portrait and classify the system's geometric character and stability behavior.

Solution. Since the eigenvalues are complex with real part greater than zero, we have an unstable spiral.

To sketch, we need to determine if our spiral is clockwise or counterclockwise and we will do this looking at the tangent to a point. Consider the point $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Plugging this into our equation yields

$$\mathbf{x} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

This means that the tangent is pointing upwards at the point $(1, 0)$. The only way for this to occur is if our spiral is moving in the counterclockwise direction. Since the real part is greater than zero, the spiral will "flow outward" from the origin. See sketch below. □



(c) Solve the given initial value problem: $\mathbf{x}' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$.

Solution. When $t = 0, x = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$:

$$\begin{pmatrix} -2 \\ 8 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

which yields the system

$$\begin{aligned} -2 &= c_1 \\ 8 &= c_1 - 2c_2 \end{aligned}$$

This system has the solution $c_1 = -2, c_2 = -5$, therefore the solution is

$$\begin{aligned} \mathbf{x} &= -2e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} - 5e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} \\ &= e^{5t} \begin{pmatrix} -2\cos 2t - 5\sin 2t \\ -2\cos 2t - 4\sin 2t - 5\sin 2t + 10\cos 2t \end{pmatrix} \\ &= e^{5t} \begin{pmatrix} -2\cos 2t - 5\sin 2t \\ 8\cos 2t - 9\sin 2t \end{pmatrix} \end{aligned}$$

□

7. Use the method of undetermined coefficients to solve the system $\mathbf{x}' = \begin{pmatrix} 4 & \frac{1}{3} \\ 9 & 6 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -3 \\ 10 \end{pmatrix} e^t$.

Solution. The equation is in standard form. We will begin by solving for the homogenous solution to

$\mathbf{x}' = \begin{pmatrix} 4 & \frac{1}{3} \\ 9 & 6 \end{pmatrix} \mathbf{x}$. Find the eigenvalues:

$$\begin{vmatrix} 4-\lambda & \frac{1}{3} \\ 9 & 6-\lambda \end{vmatrix} = (4-\lambda)(6-\lambda) - 3 = \lambda^2 - 10\lambda + 21 = (\lambda - 3)(\lambda - 7) = 0$$

Therefore the eigenvalues are $\lambda_1 = 3, \lambda_2 = 7$.

Now use the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvectors. When $\lambda_1 = 3$ we have the system

$$\left(\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 9 & 3 & 0 \end{array} \right)$$

Pick whichever row you want. The first row says that $x_1 = -\frac{1}{3}x_2$. Setting $x_2 = 1$ gives us the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}.$$

When $\lambda_2 = 7$ we have the system

$$\left(\begin{array}{cc|c} -3 & \frac{1}{3} & 0 \\ 9 & -1 & 0 \end{array} \right)$$

Pick whichever row you want. The second row says that $9x_1 = x_2$ or that $x_1 = \frac{1}{9}x_2$. Setting $x_2 = 1$ gives us the eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix}.$$

Therefore the homogeneous solution is

$$\mathbf{x}_h = c_1 e^{3t} \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix}.$$

Now we solve for the particular solution. Since $\mathbf{g}(t) = \begin{pmatrix} -3 \\ 10 \end{pmatrix} e^t$, we will guess

$$\mathbf{x}_p = \mathbf{a}e^t = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t.$$

Find the first derivative:

$$\mathbf{x}_p' = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t$$

and plug this into our equation:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{pmatrix} 4 & \frac{1}{3} \\ 9 & 6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} -3 \\ 10 \end{pmatrix} e^t$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{pmatrix} 4a_1 + \frac{1}{3}a_2 \\ 9a_1 + 6a_2 \end{pmatrix} e^t + \begin{pmatrix} -3 \\ 10 \end{pmatrix} e^t$$

Group all the like terms together:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{pmatrix} 4a_1 + \frac{1}{3}a_2 - 3 \\ 9a_1 + 6a_2 + 10 \end{pmatrix} e^t$$

We then have the system of equations

$$\begin{cases} a_1 &= 4a_1 + \frac{1}{3}a_2 - 3 \\ a_2 &= 9a_1 + 6a_2 + 10 \end{cases} \Rightarrow \begin{cases} 3a_1 + \frac{1}{3}a_2 &= 3 \\ 9a_1 + 5a_2 &= -10. \end{cases}$$

The solution to this system is $a_1 = \frac{55}{36}, a_2 = -\frac{19}{4}$. Therefore

$$\mathbf{x}_p = \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^t$$

and the general solution is thus

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p = c_1 e^{3t} \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^t.$$

□

8. Consider the autonomous system

$$\begin{aligned} x' &= y - x^2 + 2 \\ y' &= x^2 - xy \end{aligned}$$

(a) Find the fixed points of the system.

Solution. Set $x', y' = 0$ to get the system of equations:

$$\begin{cases} y - x^2 + 2 & = 0 \\ x^2 - xy & = 0 \end{cases}$$

The second row factors into

$$x(x - y) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = y.$$

If $x = 0$, then from the first equation we get that $y + 2 = 0$ which is $y = -2$. Therefore $(0, -2)$ is a fixed point.
If $x = y$, then from the first equation we have

$$y - y^2 + 2 = 0 \quad \Rightarrow \quad y^2 - y - 2 = 0 \quad \Rightarrow \quad (y-2)(y+1) = 0 \quad \Rightarrow \quad y = 2 \text{ or } y = -1$$

If $y = 2$, then $x = 2$ and $(2, 2)$ is a fixed point. If $y = -1$, then $x = -1$ and $(-1, -1)$ is a fixed point.
Therefore the fixed points are $(0, -2), (2, 2), (-1, -1)$.

□

(b) Write the Jacobian J for the system above.

Solution.

$$J = \begin{pmatrix} \frac{\partial}{\partial x}(y - x^2 + 2) & \frac{\partial}{\partial y}(y - x^2 + 2) \\ \frac{\partial}{\partial x}(x^2 - xy) & \frac{\partial}{\partial y}(x^2 - xy) \end{pmatrix} = \begin{pmatrix} -2x & 1 \\ 2x - y & -x \end{pmatrix}$$

□

(c) For each of your fixed points in part (a), evaluate the Jacobian J you found in part (b) and use it to classify the type and stability of that fixed point.

Solution. At $(0, -2)$, $J(0, -2) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Find the eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 2 = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{2}$$

Since the eigenvalues are real, distinct, and opposite sign, we have an unstable saddle.

At $(2,2)$, $J(2,2) = \begin{pmatrix} -4 & 1 \\ 2 & -2 \end{pmatrix}$. Find the eigenvalues:

$$\begin{vmatrix} -4-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = (-4-\lambda)(-2-\lambda) - 2 = \lambda^2 + 6\lambda + 6 = 0$$

Plugging this into the quadratic formula we have $\lambda = -3 \pm \frac{\sqrt{10}}{2}$. Since the eigenvalues are real, distinct, and both negative, we have a stable node.

At $(-1,-1)$, $J(-1,-1) = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$. Find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda+1) = \lambda^2 - 3\lambda + 3 = 0$$

Plugging this into the quadratic formula we have $\lambda = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}$. Since the eigenvalues are complex and the real part is greater than zero, we have an unstable spiral. □