Solutions to Practice Problems II

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Answers

This page contains answers only. Detailed solutions are on the following pages.

1. (a) $y = e^{-2x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$
   (b) $y = c_1 + c_2 e^x + c_3 x e^x$
   (c) $u = c_1 e^{-\frac{1}{2}t} + c_2 e^{3t}$
   (d) $r = c_1 e^s + c_2 e^{-s} + c_3 \cos s + c_4 \sin s$
2. $W = 2e^{6x}$
3. (a) $y_2 = x^2 \ln x$
   (b) $W = x^3$
   (c) $y = c_1 x^2 + c_2 x^2 \ln x$
4. $y_2 = -t^{1/2}$
5. (a) $y_p = At^2 + Bt + C$
   (b) Answers will vary
   (c) $y_p = At^2 + Bt + C + Ee^{-2t}$
   (d) $y_p = Ae^{3t} \sin 4t + Be^{3t} \cos 4t$
   (e) Answers will vary
   (f) $y_p = A + (Bt^2 + Ct + E)e^{-t}$
   (g) $y_p = A \cos 2t + B \sin 2t$
   (h) Answers will vary
   (i) $y_p = (At + B) + (Ct + E) \sin t + (Ft + G) \cos t$
6. $y = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2} \sin 2t - \frac{1}{2} \cos 2t$
7. $y = 2e^{-2t} + 9te^{-2t} + \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right) e^{-2t}$
8. $y = c_1 \cos t + c_2 \sin t - \cos t \ln |\sec t + \tan t|$.
9. $y = c_1 \cos(\ln t) + c_2 \sin(\ln t) + \ln |\cos(\ln t)|(\cos(\ln t)) + \ln |t|(\sin(\ln t))$

Detailed Solutions

1. Find the general solution of the following homogeneous higher-order differential equations:

   (a) $y'' + 4y' + 7y = 0$ (use x as the independent variable)

   Solution. The characteristic equation is $r^2 + 4r + 7 = 0$. Using the quadratic equation, we see the roots are

   $$r = \frac{-4 \pm \sqrt{4^2 - 4(1)(7)}}{2} \quad \Rightarrow \quad r = \frac{-4 \pm i2\sqrt{3}}{2} = -2 \pm i\sqrt{3}.$$

   We have two complex roots, hence the solution to the differential equation is

   $$y = e^{-2x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

   □

   (b) $y^{(3)} + 2y'' + y' = 0$ (use x as the independent variable)

   Solution. The characteristic equation is $r^3 + 2r^2 + r = 0$ which factors as

   $$r(r^2 + 2r + 1) = 0 \quad \Rightarrow \quad r(r+1)^2 = 0.$$

   Therefore the roots are $r = 0$ multiplicity 1, $r = -1$ multiplicity 2. The solution to the differential equation is therefore

   $$y = c_1 e^{0x} + c_2 e^{1x} + c3xe^{1x} \quad \text{or} \quad y = c_1 + c_2 e^x + c_3 x e^x.$$

   □

   (c) $2\dfrac{d^2u}{dt^2} - 5\dfrac{du}{dt} - 3u = 0$

   Solution. The characteristic equation is $2r^2 - 5r - 3 = 0$, which factors as

   $$(2r+1)(r-3) = 0.$$

   The roots are $r = -\frac{1}{2}, r = 3$ each with multiplicity 1. The solution to the differential equation is therefore

   $$u = c_1 e^{-\frac{1}{2}t} + c_2 e^{3t}.$$

   □

   (d) $\dfrac{d^4r}{ds^4} - r = 0$

   Solution. The characteristic equation is $m^4 - 1 = 0$ (we cannot use r as our characteristic equation variable since it is being used in the differential equation). This factors as

   $$(m^2 - 1)(m^2 + 1) = 0 \quad \Rightarrow \quad (m-1)(m+1)(m-i)(m+i) = 0.$$

   The roots are $m = 1, -1, \pm i$. The solution to the differential equation is

   $$r = c_1 e^s + c_2 e^{-s} + e^{0s}(c_3 \cos s + c_4 \sin s) \quad \Rightarrow \quad r = c_1 e^s + c_2 e^{-s} + c_3 \cos s + c_4 \sin s.$$

   □

2. Calculate $W(y_1, y_2, y_3)$ where $y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x}$.

Solution.

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ (e^x)' & (e^{2x})' & (e^{3x})' \\ (e^x)'' & (e^{2x})'' & (e^{3x})'' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= e^x \begin{vmatrix} 2e^{2x} & 3e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix} - e^x \begin{vmatrix} e^{2x} & e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix} + e^x \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$$

$$= e^x(18e^{5x} - 12e^{5x}) - e^x(9e^{5x} - 4e^{5x}) + e^x(3e^{5x} - 2e^{3x})$$

$$= e^x(6e^{5x} - 5e^{5x} + e^{5x})$$

$$= 2e^{6x}$$

□

3. (a) The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Use the method of reduction of order to find a second solution y_2 to the differential equation on the interval $(0, \infty)$.

Proof. We need to write the differential equation in standard form by dividing through by x^2:

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0 \qquad (1)$$

Let $y_2 = vy_1$ where v is a function of x. Since $y_1 = x^2$, then $y_2 = x^2v$. We wish to plug this into the equation above, so we need to find y_2', y_2'' using the product rule:

$y_2 = x^2v$

$y_2' = (x^2)'v + x^2v' = 2xv + x^2v'$

$y_2'' = (2xv)' + (x^2v')' = (2x)'v + 2xv' + (x^2)'v' + x^2v'' = 2v + 2xv' + 2xv' + x^2v'' = 2v + 4xv' + x^2v''$

Substitute these into the standard form equation (5) above:

$$2v + 4xv' + x^2v'' - \frac{3}{x}(2xv + x^2v') + \frac{4}{x^2}(x^2v) = 0$$

$$\Rightarrow \quad 2v + 4xv' + x^2v'' - 6xv - 3xv' + 4v = 0$$

$$\Rightarrow \quad x^2v'' + xv' = 0 \qquad (2)$$

Now let $w = v'$, then $w = v''$ and (6) becomes

$$x^2w' + xw = 0 \quad \Rightarrow \quad x^2\frac{dw}{dx} = -xw.$$

This is a separable equation. Separate the variables:

$$\frac{dw}{w} = -\frac{1}{x}dx$$

Integrate both sides:

$$\int \frac{dw}{w} = -\int \frac{1}{x}dx \quad \Rightarrow \quad \ln w = -\ln x \quad \Rightarrow \quad \ln w = \ln\left(\frac{1}{x}\right)$$

Solve for w:

$$w = \frac{1}{x}.$$

But, $w = v'$, and so we have

$$v' = \frac{1}{x} \quad \Rightarrow \quad v = \int \frac{1}{x}dx = \ln x$$

Therefore

$$y_2 = x^2v \quad \Rightarrow \quad y_2 = x^2 \ln x.$$

□

(b) Show that y_1 and y_2 form a fundamental set of solutions to the differential equation.

Solution. We need to show that $W(y_1, y_2) \neq 0$ in the interval $(0, \infty)$.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} x^2 & x^2 \ln x \\ (x^2)' & (x^2 \ln x)' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} \\ &= x^2(2x \ln x + x) - 2x(x^2 \ln x) \\ &= x^3 \end{aligned}$$

$W = x^3$ which is nonzero since we are in the interval $(0, \infty)$. □

(c) Write the general solution of the differential equation using y_1 and y_2.

Solution. The general solution to the differential equation is $y = c_1 y_1 + c_2 y_2$, which is

$$y = c_1 x^2 + c_2 x^2 \ln x.$$

□

4. Use reduction of order to find a second solution y_2 to the differential equation

$$4t^2 y'' + y = 0$$

given that $y_1 = t^{1/2} \ln t$ is a solution.

Solution. Let $y_2 = v y_1 = v(t^{1/2} \ln t)$. We need to find y_2', y_2'':

$$\begin{aligned} y_2' &= v'(t^{1/2} \ln t) + v(t^{1/2} \ln t))' = v'(t^{1/2} \ln t) + v\left(\frac{1}{2} t^{-1/2} \ln t + t^{-1/2}\right) \\ &= v'(t^{1/2} \ln t) + v\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) \\ y_2'' &= v''(t^{1/2} \ln t) + v'(t^{1/2} \ln t)' + v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) + v\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right)' \\ &= v''(t^{1/2} \ln t) + 2v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) + v\left(-\frac{1}{2} t^{-3/2}\left(\frac{1}{2} \ln t + 1\right) + t^{-1/2} \cdot \frac{1}{2t}\right) \\ &= v''(t^{1/2} \ln t) + 2v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) + v\left(-\frac{1}{4} t^{-3/2} \ln t\right) \end{aligned}$$

Now plug these into the original equation:

$$\begin{aligned} 4t^2 y'' + y &= 4t^2\left(v''(t^{1/2} \ln t) + 2v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) + v\left(-\frac{1}{4} t^{-3/2} \ln t\right)\right) + v(t^{1/2} \ln t) \\ &= 4v'' t^{5/2} \ln t + 8t^2 v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) - t^2 v\left(t^{-3/2} \ln t\right) + v(t^{1/2} \ln t) \\ &= 4v'' t^{5/2} \ln t + 8t^2 v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) \end{aligned}$$

Now set this equal to 0:

$$4v'' t^{5/2} \ln t + 8t^2 v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) = 0$$

Let $w = v'$, then $w' = v''$:

$$4w' t^{5/2} \ln t + 8t^2 w\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) = 0$$

This is a separable equation. We can rewrite w' as $\frac{dw}{dx}$ and get w's on one side, t's on the other:

$$4\frac{dw}{dt}t^{5/2}\ln t = -8t^2 w\left(t^{-1/2}\left(\frac{1}{2}\ln t + 1\right)\right)$$

$$\frac{dw}{w} = \frac{-2t^2\left(t^{-1/2}\left(\frac{1}{2}\ln t + 1\right)\right)}{t^{5/2}\ln t}$$

Simplify the right hand side:

$$\frac{-t^{3/2}\ln t - 2t^{3/2}}{t^{5/2}\ln t} = -\frac{1}{t} - \frac{2}{t\ln t}$$

Integrate both sides:

$$\int \frac{dw}{w} = \int\left(-\frac{1}{t} - \frac{2}{t\ln t}\right)dt$$

On the right hand side you will need to use the substitution $a = \ln t, da = \frac{1}{t}dt$:

$$\ln|w| = -\ln|t| - 2\ln|\ln t|$$

Now solve for w. But first we need to combine the log terms on the right hand side:

$$\ln|w| = \ln\left|\frac{1}{t}\right| + \ln\left|\frac{1}{(\ln t)^2}\right|$$

$$\ln|w| = \ln\left|\frac{1}{t(\ln t)^2}\right|$$

Take the exponential of both sides:

$$w = \frac{1}{t(\ln t)^2}$$

But, $w = v'$, so

$$v = \int \frac{1}{t(\ln t)^2}dt$$

and using the substitution $a = \ln t, da = \frac{1}{t}dt$,

$$v = \int \frac{1}{a^2}da = -\frac{1}{a} = -\frac{1}{\ln t.}$$

Therefore

$$y_2 = v_2 t^{1/2}\ln t = -\frac{1}{\ln t}\cdot t^{1/2}\ln t = -t^{1/2}.$$

□

5. Suppose you are solving the equation $y'' + P(t)y' + Q(t)y = g(t)$, where $P(t)$ and $Q(t)$ are constants, using the method of undetermined coefficients. Complete the table below. Assume $g(t)$ has no function in common with the homogeneous solution y_h. List your solutions for (a) – (i).

$g(t)$	Form of y_p	$g(t)$	Form of y_p
$3t^2 - 2$	(a)	$1 - t^2e^{-t}$	(f)
(b)	Ae^{5t}	$3\cos 2t$	(g)
$6t^2 + 2 - 12e^{-2t}$	(c)	(h)	$At + B$
$e^{3x}\sin 4x$	(d)	$4t(1 + 3\sin t)$	(i)
(e)	$(At + B)e^{-3t}$		

Solution. (a) $g(t)$ is a second degree polynomial, so we must guess a second degree polynomial: $y_p = At^2 + Bt + C$

(b) y_p is an exponential function, so g must have also been an exponential function: $g = 100e^{5t}$ (answers will vary)

(c) $g(t)$ is a second degree polynomial added to an exponential function, so we must guess a second degree polynomial added to an exponential function: $y_p = At^2 + Bt + C + Ee^{-2t}$.

(d) $g(t)$ is an exponential function multiplied by a sine function, so we must also guess an exponential function multiplied by a sine function (we will also have to include a cosine function since sine and cosine come in pairs): $y_p = Ae^{3t}\sin 4t + Be^{3t}\cos 4t$

(e) y_p is a first degree polynomial multiplied by an exponential function, so g must have been a first degree polynomial multiplied by an exponential function: $g = 18te^{-3t}$ (answers will vary)

(f) $g(t)$ is a constant term added to a second degree polynomial multiplied by an exponential function, so we guess the same thing: $y_p = A + (Bt^2 + Ct + E)e^{-t}$

(g) $g(t)$ is a cosine function, so we must also guess a cosine function (we will also have to include a sine function since sine and cosine come in pairs): $y_p = A\cos 2t + B\sin 2t$

(h) y_p is a first degree polynomial, so g must have also been a first degree polynomial: $g = -t + 3$ (answers will vary)

(i) Distribute first:

$$g = 4t + 12t\sin t$$

g is a first degree polynomial added to a first degree polynomial multiplied by a sine function, so we must also guess the same thing (we will also have to include a cosine function since sine and cosine come in pairs): $y_p = At + B + (Ct + E)\sin t + (Ft + G)\cos t$

□

6. Solve the given differential equation using the method of undetermined coefficients.

$$y'' + 2y' = 3 + 4\sin 2t.$$

Solution. We first find the homogeneous solution:

$$y'' + 2y' = 0$$

has the characteristic equation

$$r^2 + 2r = 0 \quad \Rightarrow \quad r(r+2) = 0 \quad r = 0, r = -2$$

therefore the homogeneous solution is $y_h = c_1 + c_2e^{-2t}$.

Now we guess a particular solution. $g(t) = 3 + 4\sin 2t$, which is a constant term added to a sine term. We must also guess the same thing, but we will need to include a cosine term as well since sine and cosine come in pairs:

$$y_p = A + B\sin 2t + C\cos 2t.$$

Now we compare with y_h to check if there are any solutions in common. Notice that y_p has a constant term in common with y_h (c_1), so we must multiply our constant A by t ("bump it up") so there is no more overlap:

$$y_p = At + B\sin 2t + C\cos 2t.$$

We compare again with y_h, and we see there are no more solutions in common, so this is our final guess.

Find y_p', y_p'':

$$y_p' = A + 2B\cos 2t - 2C\sin 2t$$
$$y_p'' = -4B\sin 2t - 4C\cos 2t$$

Substitute these values into our equation above:

$$\begin{aligned} y'' + 2y' &= -4B\sin 2t - 4C\cos 2t + 2(A + 2B\cos 2t - 2C\sin 2t) \\ &= -4B\sin 2t - 4C\cos 2t + 2A + 4B\cos 2t - 4C\sin 2t \end{aligned}$$

Set this equal to $g(t)$:

$$-4B\sin 2t - 4C\cos 2t + 2A + 4B\cos 2t - 4C\sin 2t = 3 + 4\sin 2t$$

Set like terms equal to each other:

$$\begin{aligned} 2A &= 3 \\ -4B - 4C &= 4 \\ -4C + 4B &= 0 \end{aligned}$$

The solution to this system is $A = \frac{3}{2}, B = -\frac{1}{2}, C = -\frac{1}{2}$, therefore the particular solution is $y_p = \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$. The general solution is $y = y_h + y_p$:

$$y = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t.$$

□

7. Solve the given initial-value problem.

$$y'' + 4y' + 4y = (3+t)e^{-2t}, \quad y(0) = 2, y'(0) = 5$$

Solution. We first find the homogeneous solution:

$$y'' + 4y' + 4y = 0$$

has the characteristic equation

$$r^2 + 4r + 4 = 0 \quad \Rightarrow \quad (r+2)^2 = 0 \quad \Rightarrow \quad r = -2 \text{ mult. } 2$$

therefore the homogeneous solution is $y_h = c_1 e^{-2t} + c_2 t e^{-2t}$.

We can use either the method of undetermined coefficients or variation of parameters to solve this equation. We will do both.

Undetermined coefficients: We need to guess our particular solution. $g(t) = (3+t)e^{-2t}$ is a first degree polynomial multiplied by an exponential term. We must also guess the same thing:

$$y_p = (At + B)e^{-2t}$$

Now we compare with y_h to check if there are any solutions in common. Notice that y_p has both e^{-2t} and te^{-2t} in common with y_h, so we must multiply y_p by t ("bump it up"):

$$y_p = t(At + B)e^{-2t}$$

We compare again with y_h, and we see that y_p has te^{-2t} in common with y_h, so we must multiply y_p again by t (" bump it up"):

$$y_p = t^2(At + B)e^{-2t}$$

We compare again with y_h, and we see there are no more solutions in common, so this is our final guess.

Find y_p', y_p'':

$$\begin{aligned} y_p &= (At^3 + Bt^2)e^{-2t} \\ y_p' &= (3At^2 + 2Bt)e^{-2t} - 2(At^3 + Bt^2)e^{-2t} \\ y_p'' &= (6At + 2B)e^{-2t} - 2(3At^2 + 2Bt)e^{-2t} - 2(3At^2 + 2Bt)e^{-2t} + 4(At^3 + Bt^2)e^{-2t} \end{aligned}$$

Substitute these values into our equation above:

$$\begin{aligned}
y'' + 4y' + 4y &= (6At + 2B)e^{-2t} - 4(3At^2 + 2Bt)e^{-2t} + 4(At^3 + Bt^2)e^{-2t} \\
&\qquad + 4\left((3At^2 + 2Bt)e^{-2t} - 2(At^3 + Bt^2)e^{-2t}\right) + 4(At^3 + Bt^2)e^{-2t} \\
&= (6At + 2B)e^{-2t} - 4(3At^2 + 2Bt)e^{-2t} + 4(At^3 + Bt^2)e^{-2t} + 4(3At^2 + 2Bt)e^{-2t} \\
&\qquad - 8(At^3 + Bt^2)e^{-2t} + 4(At^3 + Bt^2)e^{-2t} \\
&= (6At + 2B)e^{-2t}
\end{aligned}$$

Set this equal to $g(t)$:

$$(6At + 2B)e^{-2t} = (3 + t)e^{-2t}$$

Set like terms equal to each other:

$$\begin{aligned}
6A &= 1 \\
2B &= 3
\end{aligned}$$

The solution to this system is $A = \frac{1}{6}, B = \frac{3}{2}$, therefore the particular solution is $y_p = \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t}$. The general solution is $y = y_h + y_p$:

$$y = c_1e^{-2t} + c_2te^{-2t} + \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t}.$$

Variation of Parameters: Let $y_1 = e^{-2t}$, $y_2 = te^{-2t}$ (these come from y_h above). Then

$$\begin{aligned}
W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-2t}(e^{-2t} - 2te^{-2t}) - (-2e^{-2t})(te^{-2t}) = e^{-4t} \\
W_1 &= \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & te^{-2t} \\ (3+t)e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = -(3+t)e^{-2t}(te^{-2t}) = -(3+t)te^{-4t} \\
W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} e^{-2t} & 0 \\ -2e^{-2t} & (3+t)e^{-2t} \end{vmatrix} = e^{-2t}(3+t)e^{-2t} = (3+t)e^{-4t}
\end{aligned}$$

Find u_1, u_2:

$$\begin{aligned}
u_1' &= \frac{W_1}{W} = \frac{-(3+t)te^{-4t}}{e^{-4t}} = -3t - t^2 \quad \Rightarrow \quad u_1 = \int(-3t - t^2)dt = -\frac{3}{2}t^2 - \frac{1}{3}t^3 \\
u_2' &= \frac{W_2}{W} = \frac{(3+t)e^{-4t}}{e^{-4t}} = 3 + t \quad \Rightarrow \quad u_2 = \int(3+t)dt = 3t + \frac{1}{2}t^2
\end{aligned}$$

The particular solution is $y_p = u_1y_1 + u_2y_2$:

$$\begin{aligned}
y_p &= \left(-\frac{3}{2}t^2 - \frac{1}{3}t^3\right)e^{-2t} + \left(3t + \frac{1}{2}t^2\right)te^{-2t} \\
&= -\frac{3}{2}t^2e^{-2t} - \frac{1}{3}t^3e^{-2t} + 3te^{-2t} + \frac{1}{2}t^3e^{-2t} \\
&= \frac{3}{2}t^2e^{-2t} + \frac{1}{6}t^3e^{-2t}
\end{aligned}$$

The general solution is $y = y_h + y_p$:

$$y = c_1e^{-2t} + c_2te^{-2t} + \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t}.$$

We aren't finished! We need to find c_1, c_2 given $y(0) = 2, y'(0) = 5$. Find y':

$$\begin{aligned}
y &= c_1e^{-2t} + c_2te^{-2t} + \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t} \\
y' &= -2c_1e^{-2t} + c_2(e^{-2t} - 2te^{-2t}) + \left(\frac{1}{2}t^2 + 3t\right)e^{-2t} - 2\left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t}
\end{aligned}$$

When $t = 0, y = 2, y' = 5$:

$$\begin{aligned} 2 &= c_1 \\ 5 &= -2c_1 + c_2 \end{aligned}$$

The solution to this system is $c_1 = 2, c_2 = 9$, therefore the general solution is

$$y = 2e^{-2t} + 9te^{-2t} + \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t}.$$

□

8. Find the general solution of the given differential equation using variation of parameters.

$$y'' + y = \tan t$$

Solution. We first find the homogeneous solution:

$$y'' + y = 0$$

has the characteristic equation

$$r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i$$

therefore the homogeneous solution is $y = c_1 \cos t + c_2 \sin t$.

Let $y_1 = \cos t, y_2 = \sin t$. Then

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \\ W_1 &= \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin t \\ \tan t & \cos t \end{vmatrix} = -\sin t \tan t = -\frac{\sin^2 t}{\cos t} \\ W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} \cos t & 0 \\ -\sin t & \tan t \end{vmatrix} = \cos t \tan t = \sin t \end{aligned}$$

Find u_1 and u_2:

$$\begin{aligned} u_1' &= \frac{W_1}{W} = -\frac{\sin^2 t}{\cos t} \\ &\Rightarrow \quad u_1 = -\int \frac{\sin^2 t}{\cos t} dt = -\int \frac{1 - \cos^2 t}{\cos t} dt = -\int (\sec t - \cos t) dt = -\ln|\sec t + \tan t| + \sin t \\ u_2' &= \frac{W_2}{W} = \sin t \\ &\Rightarrow \quad u_2 = \int \sin t dt = -\cos t \end{aligned}$$

The particular solution is $y_p = u_1 y_1 + u_2 y_2$:

$$y_p = (-\ln|\sec t + \tan t| + \sin t)\cos t + \sin t \cos t = -\cos t \ln|\sec t + \tan t|$$

The general solution is $y = y_h + y_p$:

$$y = c_1 \cos t + c_2 \sin t - \cos t \ln|\sec t + \tan t|.$$

□

9. $y_1 = \cos(\ln t), y_2 = \sin(\ln t)$ are independent solutions of the equation $t^2y'' + ty' + y = 0$. Find the general solution of the equation

$$t^2y'' + ty' + y = \sec(\ln t).$$

Solution. Use variation of parameters to solve. Write the equation in standard form:

$$y'' + \frac{1}{t}y' + \frac{1}{t^2}y = \frac{\sec(\ln t)}{t^2}$$

We are given that the homogeneous solution is

$$y_h = c_1 \cos(\ln t) + c_2 \sin(\ln t).$$

Let $y_1 = \cos(\ln t), y_2 = \sin(\ln t)$. Then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\ln t) & \sin(\ln t) \\ -\frac{\sin(\ln t)}{t} & \frac{\cos(\ln t)}{t} \end{vmatrix} = \frac{\cos^2(\ln t)}{t} + \frac{\sin^2(\ln t)}{t} = \frac{1}{t}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin(\ln t) \\ \frac{\sec(\ln t)}{t^2} & \frac{\cos(\ln t)}{t} \end{vmatrix} = -\frac{\sin(\ln t)\sec(\ln t)}{t^2} = -\frac{\tan(\ln t)}{t^2}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} \cos(\ln t) & 0 \\ -\frac{\sin(\ln t)}{t} & \frac{\sec(\ln t)}{t^2} \end{vmatrix} = \frac{1}{t^2}$$

Find u_1 and u_2:

$$u_1' = \frac{W_1}{W} = \frac{-\frac{\tan(\ln t)}{t^2}}{\frac{1}{t}} = -\frac{\tan(\ln t)}{t}$$

$$\Rightarrow \quad u_1 = -\int \frac{\tan(\ln t)}{t}dt = -\int \tan a da = -\int \frac{\sin a}{\cos a}da = \int \frac{1}{b}db = \ln|b| = \ln|\cos(\ln t)|$$

$$u_2' = \frac{W_2}{W} = \frac{\frac{1}{t^2}}{\frac{1}{t}} = \frac{1}{t}$$

$$\Rightarrow \quad u_2 = \int \frac{1}{t}dt = \ln|t|$$

For u_1 we used the substitutions $a = \ln t, da = \frac{1}{t}dt$, and $b = \cos a, db = -\sin a da$. The particular solution is $y_p = u_1y_1 + u_2y_2$:

$$y = \ln|\cos(\ln t)|(\cos(\ln t)) + \ln|t|(\sin(\ln t)).$$

The general solution is $y = y_h + y_p$:

$$y = c_1 \cos(\ln t) + c_2 \sin(\ln t) + \ln|\cos(\ln t)|(\cos(\ln t)) + \ln|t|(\sin(\ln t)).$$

□