Solutions to Practice Problems I

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Answers

This page contains answers only. Detailed solutions are on the following pages.

1. (a) Answers will vary
   (b) Answers will vary
2. See detailed solution
3. See detailed solution
4. $y = \sin\left(\frac{1}{2}x^2 + \frac{\pi}{6}\right)$
5. $y + 2\ln|y-1| = x + 5\ln|x-3| + C$
6. $y = \dfrac{-\frac{1}{2}\cos 2x + C}{xe^x}$
7. $y^2 \sin x - x^3 y - x^2 + y\ln y - y = 0$

Detailed Solutions

1. Give an example of a third (3rd) order
   (a) nonlinear differential equation
   (b) linear differential equation

   Solution. Recall a third order linear differential equation is given by

   $$a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

   where a_3, a_2, a_1, a_0 are a functions of x, not of y. One way to make a nonlinear differential equation is to have functions of y in front of the derivative terms. Some examples are below:

   (a) $\sqrt{xy}y''' - e^x y' = x^2$ is a nonlinear differential equation because of the term $\sqrt{xy}y'''$.
   (b) $y''' - 2xy'' + \frac{1}{\ln x}y' - \sqrt{e^x + 4}y = 0$ is a linear differential equation because all of the functions in front of the derivative terms are functions of x alone.

   □

2. Verify $P = \dfrac{ce^t}{1+ce^t}$ is a solution to the differential equation $\dfrac{dP}{dt} = P(1-P)$.

   Solution. We need to substitute P into the equation above and show that the left hand side equations the right hand side.
   Left hand side: (use the quotient rule or product rule for the derivative)

   $$\frac{dP}{dt} = \frac{(1+ce^t)(ce^t)' - ce^t(1+ce^t)'}{(1+ce^t)^2} = \frac{(1+ce^t)(ce^t) - ce^t(ce^t)}{(1+ce^t)^2} = \frac{ce^t}{(1+ce^t)^2}$$

   Right hand side:

   $$P(1-P) = \frac{ce^t}{1+ce^t}\left(1 - \frac{ce^t}{1+ce^t}\right) = \frac{ce^t}{1+ce^t}\left(\frac{1+ce^t}{1+ce^t} - \frac{ce^t}{1+ce^t}\right) = \frac{ce^t}{1+ce^t}\cdot\frac{1}{1+ce^t} = \frac{ce^t}{(1+ce^t)^2}$$

   The left hand side equals the right hand side, hence $P = \dfrac{ce^t}{1+ce^t}$ is a solution to the differential equation $\dfrac{dP}{dt} = P(1-P)$. □

3. Sketch the direction field for the equation

   $$\frac{dy}{dx} = 1 - xy$$

   for $-4 \le x, y \le 4$. Sketch the approximate solution curves that pass through the indicated points (you should have one curve per given point):

   (a) $y(0) = 0$

(b) $y(-1) = 0$

(c) $y(2) = 2$

(d) $y(0) = -4$

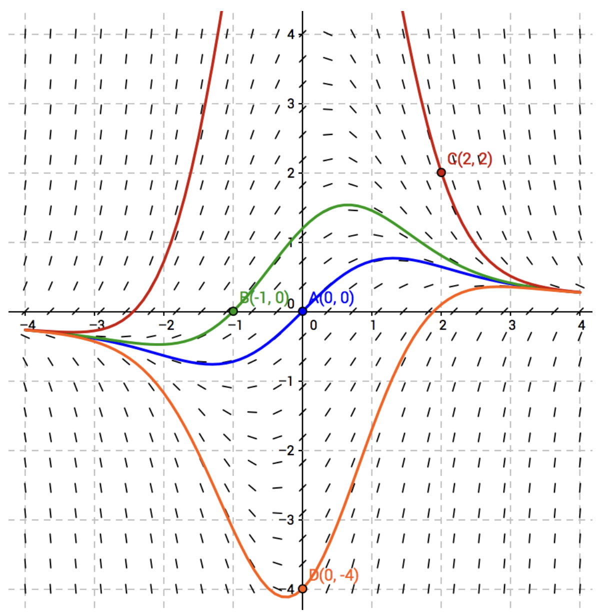
Solution. Choose points for x, y and plot the appropriate slope. For example, at the point $(0, 0)$, we have a slope of 1

$$\frac{dy}{dx} = 1 - 0 \cdot 0 = 1.$$

Recommendation for how to plot:

- Set $x = 0$ and plug in points along the y-axis.
- Set $y = 0$ and plug in points along the x-axis.
- Plug in other points.

See the graph below. Your sketch doesn't need to be exact, but should be similar. □



4. Solve the following initial value problem:

$$\frac{dy}{dx} = x\sqrt{1 - y^2}, \qquad y(0) = \frac{1}{2}.$$

Solution. This is a separable equation. We can separate the variables as follows:

$$\frac{dy}{\sqrt{1 - y^2}} = x dx$$

Integrate both sides:

$$\int \frac{dy}{\sqrt{1-y^2}} = \int x dx$$
$$\Rightarrow \quad \sin^{-1}(y) = \frac{1}{2}x^2 + C \tag{1}$$

Here we will plug in the initial value (you can also solve for y then plug in the initial value if you wish). When $x = 0, y = \frac{1}{2}$:

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}(0)^2 + C \quad \Rightarrow \quad C = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Therefore (1) becomes

$$\sin^{-1}(y) = \frac{1}{2}x^2 + \frac{\pi}{6}$$

This is the general solution in implicit form. If we were asked to solve for explicit form, then we would need to solve for y:

$$y = \sin\left(\frac{1}{2}x^2 + \frac{\pi}{6}\right).$$

□

5. Solve the following separable equation:

$$\frac{dy}{dx} = \frac{xy + 2y - x - 2}{xy - 3y + x - 3}.$$

Solution. We need to factor the right hand side:

$$\frac{dy}{dx} = \frac{y(x+2) - (x+2)}{y(x-3) + (x-3)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{(x+2)(y-1)}{(x-3)(y+1)}$$

Separate the variables as follows:

$$\frac{y+1}{y-1}dy = \frac{x+2}{x-3}dx$$

Integrate both sides:

$$\int \frac{y+1}{y-1}dy = \int \frac{x+2}{x-3}dx \tag{2}$$

You will need to use substitution or long division to evaluate these integrals. I will do substitution.

Left hand side: Let $u = y - 1$, then $du = dy$ and $y = u + 1$. We substitute these into the integral:

$$\int \frac{y+1}{y-1}dy = \int \frac{u+1+1}{u}du = \int \left(1 + \frac{2}{u}\right) du = u + 2\ln|u| = y - 1 + 2\ln|y-1|$$

Right hand side: Let $v = x - 3$, then $dv = dx$ and $x = v + 3$. We substitute these into the integral:

$$\int \frac{x+2}{x-3} dx = \int \frac{v+3+2}{v} du = \int \left(1 + \frac{5}{v}\right) du = v + 5\ln|v| = x - 3 + 5\ln|x-3|$$

Therefore the equation (2) becomes

$$y - 1 + 2\ln|y-1| = x - 3 + 5\ln|x-3| + C \quad \Rightarrow \quad y + 2\ln|y-1| = x + 5\ln|x-3| + C$$

This would be too difficult to solve for y, so we will leave it in implicit form.

□

6. Solve the following first order linear equation:

$$xy' + (1+x)y = e^{-x}\sin 2x.$$

Solution. Write the equation in standard form by dividing the equation on both sides by x:

$$y' + \left(\frac{1+x}{x}\right) y = \frac{e^{-x}\sin 2x}{x}$$

Now we wish to find an integrating factor. From the equation above, we see that

$$P(x) = \frac{1+x}{x} = \frac{1}{x} + 1$$

(remember that $P(x)$ is function in front of the y term in the differential equation). Therefore

$$\mu(x) = \exp\left(\int P(x)dx\right) = \exp\left(\int \left(\frac{1}{x} + 1\right) dx\right) = e^{\ln x + x} = e^{\ln x}e^{x} = xe^{x}$$

Multiply the standard form equation above by $\mu(x)$:

$$xe^{x}\left[y' + \left(\frac{1+x}{x}\right) y = \frac{e^{-x}\sin 2x}{x}\right]$$

$$\Rightarrow \quad xe^{x}y' + (1+x)e^{x}y = \sin 2x \tag{3}$$

The left hand side should collapse down to the derivative of the product $\mu(x)y$, that is,

$$\frac{d}{dx}\left(xe^{x}y\right) = \sin 2x. \tag{4}$$

Don't believe it? You can always check:

$$\frac{d}{dx}\left(xe^{x}y\right) = xe^{x}y' + (xe^{x})'y = xe^{x}y' + (x'e^{x} + x(e^{x})')y = xe^{x}y' + (e^{x} + xe^{x})y$$

which is the same as in (3).

Now looking back at equation (4), we integrate both sides:

$$xe^x y = \int \sin 2x dx \quad \Rightarrow \quad xe^x y = -\frac{1}{2}\cos 2x + C$$

Solve for y:

$$y = \frac{-\frac{1}{2}\cos 2x + C}{xe^x}$$

□

7. Solve the initial value problem

$$(y^2 \cos x - 3x^2 y - 2x)dx + (2y \sin x - x^3 + \ln(y))dy = 0, \qquad y(0) = e.$$

Solution. This equation is of the form $Mdx + Ndy = 0$. Let's check to see if the equation is exact; that is, we need to see if $M_y = N_x$:

$$M = y^2 \cos x - 3x^2 y - 2x \quad \Rightarrow \quad M_y = 2y\cos x - 3x^2$$

$$N = 2y \sin x - x^3 + \ln(y) \quad \Rightarrow \quad N_x = 2y\cos x - 3x^2$$

$M_y = N_x$ so this equation is exact!

We want to find an $f(x, y)$ such that $\frac{\partial f}{\partial x} = M$, so we will integrate M with respect to x:

$$\begin{aligned} f(x,y) &= \int M dx + g(y) \\ &= \int (y^2 \cos x - 3x^2 y - 2x)dx + g(y) \\ &= y^2 \sin x - x^3 y - x^2 + g(y) \end{aligned}$$

We want $\frac{df}{dy} = N$, so we will differentiate f with respect to y and set it equal to N:

$$\begin{aligned} \frac{\partial f}{\partial y} = N \quad &\Rightarrow \quad \frac{\partial}{\partial y}(y^2 \sin x - x^3 y - x^2 + g(y)) = 2y \sin x - x^3 + \ln(y) \\ &\Rightarrow \quad 2y \sin x - x^3 + g'(y) = 2y \sin x - x^3 + \ln y \end{aligned}$$

Solve for $g'(y)$:

$$g'(y) = \ln y$$

and integrate:

$$g(y) = \int \ln y dy.$$

You will need to use integration by parts to evaluate this integral. Try $u = \ln y, dv = dy$. Then $du = \frac{1}{y}dy, v = y$ and we have

$$g(y) = \int \ln y dy = y \ln y - \int y \cdot \frac{1}{y} dy = y \ln y - \int dy = y \ln y - y$$

Substituting this into our equation above for f, we have

$$f(x,y) = y^2 \sin x - x^3 y - x^2 + y \ln y - y.$$

Now set $f(x,y) = C$:

$$y^2 \sin x - x^3 y - x^2 + y \ln y - y = C$$

This is the general solution to the differential equation. Now we need to plug in the initial value $y(0) = e$. When $x = 0, y = e$:

$$e^2 \sin 0 - (0)^3 \cdot e - (0)^2 + e \ln e - e = C \quad \Rightarrow \quad C = 0$$

Therefore the solution is $y^2 \sin x - x^3 y - x^2 + y \ln y - y = 0$. □