Exact Equations

def. An equation of the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is said to be **exact**

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Goal: Find a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = M(x,y)$, $\frac{\partial f}{\partial y} = N(x,y)$.

Steps to solve an exact equation:

① Verify $M_y = N_x$ (here $M_y = \frac{\partial M}{\partial y}$, $N_x = \frac{\partial N}{\partial x}$).

② We want $\frac{\partial f}{\partial x} = M$.

Integrate M w.r.t. x: $f = \int M(x,y)\,dx + g(y)$

③ we want $\frac{\partial f}{\partial y} = N$. Differentiate f in ② w.r.t. y:

$$\frac{\partial}{\partial y}\left(\int M(x,y)\,dx + g(y)\right) = N(x,y)$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + g'(y) = N(x,y)$$

④ Solve for $g'(y)$ and integrate to get g.

⑤ Plug g(y) back into ②. Write solution as $f(x,y) = C$.

First we should review partial derivatives and integration...

Ex $f(x,y) = x^4 + e^{xy} + \ln y$

$$\frac{\partial f}{\partial x} = 4x^3 + e^{xy}\frac{\partial}{\partial x}(xy) = 4x^3 + ye^{xy}$$

this means y acts as a constant

$$\frac{\partial f}{\partial y} = e^{xy}\frac{\partial}{\partial y}(xy) + \frac{1}{y} = xe^{xy} + \frac{1}{y}.$$

means x acts as constant

Ex Let $M = 4x^3 + ye^{xy} \leftarrow \frac{\partial f}{\partial x}$ from previous example

$N = xe^{xy} + \frac{1}{y} \leftarrow \frac{\partial f}{\partial y}$ from previous example

If we integrate M w.r.t. x (partial integration):

$$\int M\,dx = \int (4x^3 + ye^{xy})\,dx = x^4 + e^{xy} + \underbrace{g(y)}$$

constants and functions of y since these disappear when we take derivative w.r.t. x

Similarly,

$$\int N\,dy = \int \left(xe^{xy} + \frac{1}{y}\right) dy = e^{xy} + \ln|y| + \underbrace{h(x)}$$

constants, functions of x.

We need these extra functions or else we lose information!

Ex Solve the equation $e^{2y} - y\cos xy + (2xe^{2y} - x\cos xy + 2y)\frac{dy}{dx} = 0.$

$N = 2xe^{2y} - x\cos xy + 2y$

(1) $M = e^{2y} - y\cos xy$

$M_y = 2e^{2y} - (y'\cos xy + y(\cos xy)')$

$= 2e^{2y} - (\cos xy + y(-\sin(xy))x)$

$= 2e^{2y} - \cos xy + xy\sin xy$ exact!

$N_x = 2e^{2y} - (x'\cos xy + x(\cos xy)')$

$= 2e^{2y} - (\cos xy + x(-\sin xy)\cdot y)$

$= 2e^{2y} - \cos xy + xy\sin xy$

(2) $\frac{\partial f}{\partial x} = M \Rightarrow f = \int M\,dx + g(y)$

$= \int (e^{2y} - y\cos xy)\,dx + g(y)$

$= xe^{2y} - \frac{y\sin xy}{y} + g(y)$

$= xe^{2y} - \sin xy + g(y)$

(3) $\frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}\left(xe^{2y} - \sin xy + g(y)\right) = 2xe^{2y} - x\cos xy + 2y$

$2xe^{2y} - \cos(xy)\cdot x + g'(y) = 2xe^{2y} - x\cos xy + 2y$

(4) Solve for $g'(y)$:

$g'(y) = 2y$

$g(y) = \int 2y\,dy$

$= y^2$

(5) $f = xe^{2y} - \sin xy + y^2 = C$

$$\boxed{xe^{2y} - \sin xy + y^2 = C}$$