Practice Midterm 1 Solutions

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1. Use the slope field plotter link in Gauchospace to check your solution.

2. (a) Not linear because of the $y^2 \sin x$ term
   (b) Not linear because of the $\frac{t}{y}$ term
   (c) Linear, we can write the equation as $y' - y = t$.
   (d) Not linear because of the $\frac{r^2}{\theta}$ term.

3. This is a separable equation. Separate the variables:

$$2ydy = (2t+5)dt$$

Integrate both sides:

$$\int 2ydy = \int (2t+5)dt$$
$$y^2 = t^2 + 5t + C$$

Now solve for C. When $t = 0, y = -2$:

$$(-2)^2 = 0^2 + 5 \cdot 0 + C \quad \Rightarrow \quad C = 4$$

Plug back in and solve for y:

$$y^2 = t^2 + 5t + 4$$
$$y = \pm\sqrt{t^2 + 5t + 4}$$

Use the initial condition to decide whether to choose + or −. When $t = 0, y = -2$, hence

$$y = -\sqrt{t^2 + 5t + 4}.$$

4. (a) This is a first order linear equation and is already in standard form. Find the integrating factor:

$$\mu = e^{\int 2dt} = e^{2t}$$

Multiply the equation by μ:

$$e^{2t}y' + 2e^{2t}y = 3e^t e^{2t}$$
$$e^{2t}y' + (e^{2t})'y = 3e^{3t}$$
$$(e^{2t}y)' = 3e^{3t}$$

Integrate both sides:

$$e^{2t}y = \int 3e^{3t}dt$$
$$e^{2t}y = e^{3t} + C$$

Solve for y:

$$y = e^t + Ce^{-2t}$$

Solve for C. When $t = 0, y = -3$:

$$-3 = e^0 + Ce^0 \quad \Rightarrow \quad C = -4$$

Plug back in:

$$y = e^t - 4e^{-2t}$$

(b) This is a first order linear equation and is already in standard form. Find the integrating factor:

$$\mu = e^{\int \frac{1}{t} dt} = e^{\ln |t|} = t$$

Multiply the equation by μ:

$$\begin{aligned} ty' + 1 \cdot y &= \frac{t}{t^2} \\ ty' + t'y &= \frac{1}{t} \\ (ty)' &= \frac{1}{t} \end{aligned}$$

Integrate both sides:

$$\begin{aligned} ty &= \int \frac{1}{t} dt \\ ty &= \ln |t| + C \\ ty &= \ln t + C \quad |t| = t \text{ since } t > 0 \end{aligned}$$

Solve for y:

$$y = \frac{\ln t}{t} + \frac{C}{t}$$

Solve for C. When $t = 1, y = 1$:

$$1 = \frac{\ln 1}{1} + \frac{C}{1} \quad \Rightarrow \quad C = 1$$

Plug back in:

$$y = \frac{\ln t}{t} + \frac{1}{t}$$

(c) This is a first order linear equation, but is not in standard form. Rewrite in standard form:

$$y' + \frac{2t}{t^2+1} y = \frac{e^t}{t^2+1}$$

Find the integrating factor:

$$\mu = e^{\int \frac{2t}{t^2+1} dt} = e^{\ln |t^2+1|} = t^2 + 1$$

Multiply the standard form equation by μ:

$$(t^2+1)y' + 2ty = e^t$$

Notice this the equation we started with!

$$\begin{aligned}(t^2+1)y' + 2ty &= e^t \\ (t^2+1)y' + (t^2+1)'y &= e^t \\ ((t^2+1)y)' &= e^t\end{aligned}$$

Integrate both sides:

$$\begin{aligned}(t^2+1)y &= \int e^t dt \\ (t^2+1)y &= e^t + C\end{aligned}$$

Solve for y:

$$y = \frac{e^t + C}{t^2+1}$$

Solve for C. When $t = 0, y = 2$:

$$2 = \frac{e^0 + C}{0+1} \quad \Rightarrow \quad C = 1$$

Plug back in:

$$y = \frac{e^t + 1}{t^2+1}$$

5. Newton's law is given by

$$m\frac{dv}{dt} = mg - k$$

where m is the mass (kg), v is the velocity (m/s), g is the gravity constant ($9.8\ m^2/s$), and k is the air resistance. From the given equation, we have

$$60\frac{dv}{dt} = 60(9.8) - 0.8v$$

Since there is no initial velocity, we have that $v = 0$ when $t = 0$. The IVP is therefore written as

$$\begin{cases} 60\frac{dv}{dt} = 60(9.8) - 0.8v \\ v(0) = 0 \end{cases}$$

There is no need to solve this equation, just set up. Note that this is a separable AND first order linear equation.

6. This is a separable equation. Rewrite y' as $\frac{dy}{dt}$ and separate the variables:

$$\frac{dy}{(1+y)(1-y)} = dt$$

We need to integrate both sides. To integrate the left hand side, we need to use partial fraction decomposition; that is, we need to find A and B such that

$$\frac{1}{(1-y)(1+y)} = \frac{A}{1+y} + \frac{B}{1-y}$$

Multiply both sides by $(1-y)(1+y)$ and simplify:

$$\begin{aligned} 1 &= A(1-y) + B(1+y) \\ 1 &= A - Ay + B + By \\ 0y + 1 &= (-A+B)y + (A+B) \end{aligned}$$

Then

$$\begin{cases} 0 &= -A + B \\ 1 &= A + B \end{cases}$$

Solving this system of equations yields $A = B = \frac{1}{2}$. Therefore

$$\frac{1}{(1-y)(1+y)} = \frac{1/2}{1+y} + \frac{1/2}{1-y}.$$

We then have the equation

$$\left(\frac{1/2}{1+y} + \frac{1/2}{1-y}\right) dy = dt$$

Integrate both sides:

$$\begin{aligned} \int \left(\frac{1/2}{1+y} + \frac{1/2}{1-y}\right) dy &= \int dt \\ \frac{1}{2}\ln|1+y| - \frac{1}{2}\ln|1-y| &= t + C \end{aligned}$$

Solve for y (use the log property that $\log M - \log N = \log(M/N)$):

$$\begin{aligned} \ln|1+y| - \ln|1-y| &= 2t + C \\ \ln\left|\frac{1+y}{1-y}\right| &= 2t + C \\ \frac{1+y}{1-y} &= Ce^{2t} \\ 1 + y &= Ce^{2t}(1-y) \\ y + Ce^{2t}y &= Ce^{2t} - 1 \\ y(1 + Ce^{2t}) &= Ce^{2t} - 1 \\ y &= \frac{Ce^{2t} - 1}{Ce^{2t} + 1} \end{aligned}$$

Solve for C. When $t = 0, y = 2$:

$$2 = \frac{Ce^0 - 1}{Ce^0 + 1} \quad \Rightarrow \quad C = -3$$

Plug back in:

$$y = \frac{-3e^{2t} - 1}{-3e^{2t} + 1} \quad \text{or} \quad y = \frac{3e^{2t} + 1}{3e^{2t} - 1}$$

7. This looks like an exact equation. $M = ye^{2y}$ and $N = xe^{2y} + 2xye^{2y} + 2$. Check $M_y = N_x$ (be careful using product rule!)

$$M_y = e^{2y} + 2ye^{2y}$$
$$N_x = e^{2y} + 2ye^{2y}$$

Since $M_y = N_x$, the equation is exact. We want $\frac{\partial f}{\partial x} = M$, integrate M with respect to x:

$$f = \int ye^{2y} dx + g(y)$$
$$f = xye^{2y} + g(y)$$

We want $\frac{\partial f}{\partial y} = N$, differentiate f above with respect to y:

$$\frac{\partial f}{\partial y}\left(xye^{2y} + g(y)\right) = xe^{2y} + 2xye^{2y} + 2$$
$$xe^{2y} + 2xye^{2y} + g'(y) = xe^{2y} + 2xye^{2y} + 2$$

Solve for $g'(y)$ and integrate to get $g(y)$:

$$g'(y) = 2$$
$$g(y) = \int 2dy$$
$$g(y) = 2y$$

Plug back into the equation for f above:

$$f = xye^{2y} + 2y$$

Set $f = C$:

$$xye^{2y} + 2y = C$$

8. This is both a first order linear equation AND an exact equation. Use whichever method you prefer, we will solve as an exact equation. Rewrite as follows:

$$-2xe^x + 8y - 6x^2 + 8xy' = 0$$

$M = -2xe^x + 8y - 6x^2$ and $N = 8x$. Check $M_y = N_x$:

$$M_y = 8$$

$$N_x = 8$$

Since $M_y = N_x$, the equation is exact. We want $\frac{\partial f}{\partial x} = M$, integrate M with respect to x (use integration by parts):

$$f = \int (-2xe^x + 8y - 6x^2)dx + g(y)$$
$$f = -2xe^x + 2e^x + 8xy - 2x^3 + g(y)$$

We want $\frac{\partial f}{\partial y} = N$, differentiate f above with respect to y:

$$\frac{\partial f}{\partial y}\left(-2xe^x + 2e^x + 8xy - 2x^3 + g(y)\right) = 8x$$
$$8x + g'(y) = 8x$$

Solve for $g'(y)$ and integrate to get $g(y)$:

$$g'(y) = 0$$
$$g(y) = C$$

Plug back into the equation for f above:

$$f = -2xe^x + 2e^x + 8xy - 2x^3 + C$$

Set $f = C$:

$$-2xe^x + 2e^x + 8xy - 2x^3 = C$$

9. Homework 3, Problem 2: A tank contains 60 kg of salt and 1000 L of water. Pure water enters a tank at the rate 12 L/min. The solution is mixed and drains from the tank at the rate 6 L/min.

   (a) What is the amount of salt in the tank initially?

   Solution. From the given information, the tank initially contained 60 kg of salt. □

   (b) Find the amount of salt in the tank after 1 hour.

   Solution. Let's set up a differential equation. Let $Q(t)$ = the amount of salt in the tank at time t. The tank initially contains 60 kg of salt, hence the initial condition is $Q(0) = 60$. The differential equation will be modeled by

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

   The rate will be of the form concentration * rate, i.e.

$$\frac{\text{amt of salt (kg)}}{\text{amt of water (L)}} * \text{rate (L/min)}$$

Let's look at the rate coming in: Pure water enters a tank at a rate of 12 L/min. There is no salt in pure water, hence the rate in is:

$$0kg/L * 12L/min = 0kg/min$$

Let's look at the rate going out: The solution drains from the tank at a rate of 6 L/min. We do not know the amount of salt in the tank (this is $Q(t)$). The amount of water in the tank is also changing with respect to time. Water is coming in at 12 L/min and a solution is leaving at 6 L/min, hence after every minute we have an extra 6 L of liquid in the tank. The amount of liquid at time t is then $1000 + 6t$. Thus the rate going out is

$$\left(\frac{Q\ kg}{1000 + 6t\ L}\right)(6L/min) = \frac{6Q}{1000 + 6t} kg/min$$

Thus our initial value problem is

$$\begin{cases} \frac{dQ}{dt} = 0 - \frac{6Q}{1000+6t} \\ Q(0) = 60 \end{cases} \quad \Rightarrow \quad \begin{cases} \frac{dQ}{dt} = -\frac{6Q}{1000+6t} \\ Q(0) = 60 \end{cases}$$

This is a separable AND first order linear equation. Use whichever method you prefer, we will solve as a separable equation. Separate the variables:

$$\frac{dQ}{-6Q} = \frac{dt}{1000 + 6t}$$

Integrate both sides:

$$\begin{aligned} -\frac{1}{6} \ln|Q| &= \frac{1}{6} \ln|1000 + 6t| + C \\ \ln|Q| &= -\ln|1000 + 6t| + C \\ Q &= C(1000 + 6t)^{-1} \end{aligned}$$

When $t = 0, Q = 60$:

$$60 = \frac{C}{1000 + 0} \quad \Rightarrow \quad C = 60 * 1000$$

Solve for Q:

$$Q = \frac{60 * 1000}{1000 + 6t}$$

Notice that t is measured in minutes. After one hour = 60 minutes, the amount of salt in the tank is

$$Q = \frac{60 * 1000}{1000 + 6 * 60} kg$$

□

(c) Find the concentration of salt in the solution in the tank as time approaches infinity.

Solution. Take the limit of the solution in (b) as $t \to \infty$:

$$\lim_{t\to\infty} \frac{60 * 1000}{1000 + 6t} = 0$$

Thus we expect for there to be no more salt in the tank after infinite amount of time. (This makes sense, we are adding pure water and draining out the salty solution). □