Practice Midterm 2 Solutions

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1. (a) Characteristic polynomial is $m^2 + 5m + 6m = 0$ which factors as

$$(m+3)(m+2) = 0.$$

Then $m_1 = -3, m_2 = -2$. The general solution is therefore $y = c_1e^{-3x} + c_2e^{-2x}$. The initial conditions yield the system of equations

$$\begin{cases} 2 &= c_1 + c_2 \\ 3 &= -3c_1 - 2c_2 \end{cases}$$

which has the solution $c_1 = -7, c_2 = 9$. Therefore $y = -7e^{-3x} + 9e^{-2x}$.

(b) Characteristic polynomial is $m^2 + 2m + 2 = 0$. We use the quadratic formula to find m:

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = -1 \pm i.$$

The general solution is therefore $y = e^{-x}(c_1 \cos x + c_2 \sin x)$. The initial conditions yield the system of equations

$$\begin{cases} 1 &= c_1 \\ -1 = -c_1 + c_2 \end{cases}$$

which has the solution $c_1 = 1, c_2 = 0$. Therefore $y = e^{-x} \cos x$.

(c) Characteristic polynomial is $m^2 - 4m + 4 = 0$ which factors as

$$(m-2)^2 = 0.$$

Then $m = 2$ with multipicity 2. The general solution is therefore $y = c_1e^{2x} + c_2xe^{2x}$. The initial conditions yield the system of equations

$$\begin{cases} 1 &= c_1 \\ 3 &= 2c_1 + c_2 \end{cases}$$

which has solution $c_1 = c_2 = 1$. Therefore $y = e^{2x} + xe^{2x}$.

2. Write the equation in standard form:

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0.$$

Set $y_2 = vt^{-1}$ and find the first and second derivatives using product rule:

$$\begin{aligned} y_2' &= v't^{-1} - vt^{-2} \\ y_2'' &= v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3} \end{aligned}$$

Plug into the differential equation and simplify:

$$v''t^{-1} - 2v't^{-2} + 2vt^{-3} + \frac{3}{2t}\left(v't^{-1} - vt^{-2}\right) + \frac{1}{2t^2}vt^{-1} = 0$$

$$v''t^{-1} - \frac{1}{2}v't^{-2} = 0$$

Set $w = v'$ and solve the separable equation:

$$w't^{-1} - \frac{1}{2}wt^{-2} = 0 \quad \Rightarrow \quad \frac{dw}{dt}\frac{1}{t} = \frac{w}{2t^2} \quad \Rightarrow \quad \int \frac{dw}{w} = \int \frac{1}{2t}dt \quad \ln|w| = \frac{1}{2}\ln|t|$$

Solve for w: $w = e^{\frac{1}{2}\ln|t|} = t^{1/2}$. Since $w = v'$, integrate w to find v:

$$v = \int w dt = \int t^{1/2} dt = \frac{2}{3}t^{3/2}$$

Earlier we said $y_2 = vt^{-1}$:

$$y_2 = \frac{2}{3}t^{3/2}t^{-1} = \frac{2}{3}t^{1/2},$$

or you can say that $y_2 = t^{1/2}$. Therefore the general solution is $y = c_1t^{-1} + c_2t^{1/2}$.

3. A only. Not B since the right hand side of the equation is not zero, and not C because of the term $2ty'$.

4. (a) The equation is already in standard form. Find the homogeneous solution to $y''-2y'+y = 0$. The characteristic equation is

   $$m^2 - 2m + 1 = 0 \quad \Rightarrow \quad (m-1)^2 = 0 \quad \Rightarrow \quad m = 1 \text{ multipicity 2.}$$

   Therefore $y_h = c_1e^t + c_2te^t$.
   We guess $y_p = Ae^{2t}$. This is a good guess since this does not have any functions in common with y_h. Find the derivatives:

   $$y_p' = 2Ae^{2t} \quad \text{and} \quad y_p'' = 4Ae^{2t}.$$

   Plugging this into the equation yields

   $$4Ae^{2t} - 2(2Ae^{2t}) + Ae^{2t} = e^{2t} \quad \Rightarrow \quad Ae^{2t} = e^{2t}$$

   The coefficients on the left hand side must be equal to the coefficients on the right hand side, hence $A = 1$. Therefore $y_p = e^{2t}$ and $y = y_h + y_p = c_1e^t + c_2te^t + e^{2t}$.

   (b) B (set $c_1 = 1, c_2 = 0$), and C (set $c_1 = c_2 = 0$).

5. (a) The equation is already in standard form. Find the homogeneous solution to $y''+2y'+y = 0$. The characteristic equation is

   $$m^2 + 2m + 1 = 0 \quad \Rightarrow \quad (m+1)^2 = 0 \quad \Rightarrow \quad m = -1 \text{ multiplicity 2.}$$

   Therefore $y_h = c_1e^{-t} + c_2te^{-t}$.

The guess $y_p = Ae^{-t}$ is not a good guess since this has functions in common with y_h, and same with $y_p = Ate^{-t}$. Therefore we must guess $y_p = At^2e^{-t}$. Find the derivatives:

$$y_p' = 2Ate^{-t} - At^2e^{-t}$$
$$y_p'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$$

Plugging this into the equation yields

$$2Ae^{-t} - 4Ate^{-t} + At^2e^{-t} + 2(2Ate^{-t} - At^2e^{-t}) + At^2e^{-t} = 2e^{-t}$$
$$2Ae^{-t} = 2e^{-t}$$

The coefficients on the left hand side must be equal to the coefficients on the right hand side, hence $A = 1$. Therefore $y_p = t^2e^{-t}$ and $y = y_h + y_p = c_1e^{-t} + c_2te^{-t} + t^2e^{2t}$.

(b) The equation is already in standard form. Find the homogeneous solution to $y'' - 3y' - 4y = 0$. The characteristic equation is

$$m^2 - 3m - 4 = 0 \quad \Rightarrow \quad (m-4)(m+1) = 0 \quad \Rightarrow \quad m = 4, m = -1$$

Therefore $y_h = c_1e^{4t} + c_2e^{-t}$.
We guess $y_p = Ae^t \cos 2t + Be^t \sin 2t = e^t(A\cos 2t + B\sin 2t)$. This is a good guess since there are no functions in common with y_h. Find the derivatives using product rule (I recommend simplifying first before taking the next derivative):

$$\begin{aligned}
y_p &= e^t(A\cos 2t + B\sin 2t) \\
y_p' &= e^t(A\cos 2t + B\sin 2t + e^t(-2A\sin 2t + 2B\cos 2t) \\
&= e^t\left((A+2B)\cos 2t + (-2A+B)\sin 2t\right) \\
y_p'' &= e^t\left((A+2B)\cos 2t + (-2A+B)\sin 2t\right) + e^t\left(-2(A+2B)\sin 2t + 2(-2A+B)\cos 2t\right) \\
&= e^t((-3A+4B)\cos 2t + (-4A-3B)\sin 2t)
\end{aligned}$$

Plugging this into the equation yields

$$(-10A - 2B)e^t\cos 2t + (2A - 10B)e^t \sin 2t = 8e^t\cos 2t$$

The coefficients on the left hand side must be equal to the coefficients on the right hand side, hence we get the system of equations

$$\begin{cases} -10A - 2B &= -8 \\ 2A - 10B &= 0 \end{cases}$$

which has the solution $A = \frac{10}{13}, B = \frac{2}{13}$. Therefore $y_p = \frac{10}{13}e^t\cos 2t + \frac{2}{13}\sin 2t$ and $y = y_h + y_p = c_1e^{4t} + c_2e^{-t} + \frac{10}{13}e^t\cos 2t + \frac{2}{13}\sin 2t$.

6. The equation is already in standard form. Find the homogeneous solution to $y'' + 2y' + 5y = 0$. The characteristic equation is

$$m^2 + 2m + 5 = 0 \quad \Rightarrow \quad m = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} = -1 \pm 2i$$

Therefore $y_h = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$.

The guess $y_p = Ae^{-t} \cos 2t + Be^{-t} \sin 2t$ is not a good guess since this is already in y_h, so we must guess $y_p = Ate^{-t} \cos 2t + Bte^{-t} \sin 2t$.

7. Write the equation in standard form:

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = t^{-1/2}$$

From Exercise 2, we already know the solution to the homogeneous equation $y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0$ is $y = c_1 t^{-1} + c_2 t^{1/2}$. Set $y_1 = t^{-1}, y_2 = t^{1/2}$. We compute the following:

$$\begin{aligned}
W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{-1} & t^{1/2} \\ -t^{-2} & \frac{1}{2}t^{-1/2} \end{vmatrix} = \frac{1}{2}t^{-3/2} - (-t^{-3/2}) = \frac{3}{2}t^{-3/2} \\
W_1 &= \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & t^{1/2} \\ t^{-1/2} & \frac{1}{2}t^{-1/2} \end{vmatrix} = -1 \\
W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} t^{-1} & 0 \\ -t^{-2} & t^{-1/2} \end{vmatrix} = t^{-3/2} \\
u_1' &= \frac{W_1}{W} = \frac{-1}{\frac{3}{2}t^{-3/2}} = -\frac{2}{3}t^{3/2} \\
u_2' &= \frac{W_2}{W} = \frac{t^{-3/2}}{\frac{3}{2}t^{-3/2}} = \frac{2}{3}
\end{aligned}$$

Find u_1 and u_2:

$$\begin{aligned}
u_1 &= \int -\frac{2}{3}t^{3/2}dt = -\frac{4}{15}t^{5/2} \\
u_2 &= \int \frac{2}{3}dt = \frac{2}{3}t
\end{aligned}$$

Therefore $y_p = u_1 y_1 + u_2 y_2 = -\frac{4}{15}t^{5/2}t^{-1} + \frac{2}{3}t \cdot t^{1/2} = \frac{2}{5}t^{3/2}$, and $y = y_h + y_p = c_1 t^{-1} + c_2 t^{1/2} + \frac{2}{5}t^{3/2}$.