# Quiz 7 Solutions

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1. Solve the IVP of the first order linear system

$$\mathbf{x}' = \begin{pmatrix} -4 & -3 \\ 6 & 5 \end{pmatrix}\mathbf{x}, \text{ with initial value } \mathbf{x}(0) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

*Solution.* Find the eigenvalues:

$$\begin{vmatrix} -4-\lambda & -3 \\ 6 & 5-\lambda \end{vmatrix} = (-4-\lambda)(5-\lambda) - (-3)(6) = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0$$

The eigenvalues are therefore $\lambda_1 = 2, \lambda_2 = -1$.

Find the corresponding eigenvectors. For $\lambda_1 = 2$, we have the system

$$\left(\begin{array}{cc|c} -4-\lambda_1 & -3 & 0 \\ 6 & 5-\lambda_1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -6 & -3 & 0 \\ 6 & 3 & 0 \end{array}\right)$$

Pick either row. The bottom row says that $6x_1 = -3x_2$ or that $x_1 = -\frac{1}{2}x_2$. Setting $x_2 = 1$ yields the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -1$, we have the system

$$\left(\begin{array}{cc|c} -4-\lambda_2 & -3 & 0 \\ 6 & 5-\lambda_2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -3 & -3 & 0 \\ 6 & 6 & 0 \end{array}\right)$$

Pick either row. The top row says that $-3x_1 = 3x_2$, or that $x_1 = -x_2$. Setting $x_2 = 1$ yields the eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The general solution is therefore

$$\mathbf{x} = c_1 e^{2t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

When $t = 0$, we have that

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} = c_1\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

which is the system

$$\left(\begin{array}{cc|c} -1/2 & -1 & -1 \\ 1 & 1 & 3 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 0 & -1/2 & 1/2 \\ 1 & 1 & 3 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 1 & 3 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 4 \end{array}\right)$$

So we have that $c_2 = -1, c_1 = 4$. Thus the solution is

$$\mathbf{x} = 4e^{2t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} - e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Note: we may have different eigenvalues and eigenvectors, but the final solution will the same (maybe just in a different form). □