Quiz 8 Solutions

Written by Victoria Kala
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1. Solve the first order linear system

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -1 & -2 \end{pmatrix} \mathbf{x}$$

Sketch the phase portrait of the system. Is the solution stable, unstable, or semistable?

Solution. Find the eigenvalues:

$$\begin{vmatrix} -\lambda & 2 \\ -1 & -2-\lambda \end{vmatrix} = -\lambda(2-\lambda) - (-2) = \lambda^2 + 2\lambda + 2 = 0$$

We use the quadratic formula:

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = ... = -1 \pm i$$

The eigenvalues are therefore $\lambda_1 = -1 + i, \lambda_2 = -1 - i$.

Find the eigenvector $\mathbf{v_1}$. For $\lambda_1 = -1 + i$, we have the system

$$\left(\begin{array}{cc|c} -\lambda & 2 & 0 \\ -1 & -2-\lambda & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -(-1+i) & 2 & 0 \\ -1 & -2-(-1+i) & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1-i & 2 & 0 \\ -1 & -1-i & 0 \end{array}\right)$$

Pick either row. The bottom row says that $-1x - 1 + (-1-i)x_2 = 0$ or that $x_1 = (-1-i)x_2$. Setting $x_2 = 1$ yields the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}.$$

We now expand $\mathbf{v_1}e^{\lambda_1 t}$ using Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ and $i^2 = -1$:

$$\begin{aligned} \mathbf{v_1}e^{\lambda_1 t} &= \begin{pmatrix} -1-i \\ 1 \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} -1-i \\ 1 \end{pmatrix} e^{-t}e^{it} = e^{-t}\begin{pmatrix} -1-i \\ 1 \end{pmatrix}(\cos t + i\sin t) \\ &= e^{-t}\begin{pmatrix} -\cos t - i\sin t - i\cos t + \sin t \\ \cos t + i\sin t \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} -\cos t + \sin t \\ \cos t \end{pmatrix} + ie^{-t}\begin{pmatrix} -\sin t - \cos t \\ \sin t \end{pmatrix} \end{aligned}$$

The general solution is $\mathbf{x} = c_1 Re(\mathbf{v_1}e^{\lambda_1 t}) + c_2 Im(\mathbf{v_1}e^{\lambda_1 t})$, or

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} -\cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -\sin t - \cos t \\ \sin t \end{pmatrix}$$

Since $e^{-t} \to 0$ as $t \to \infty$, the solution is **stable**. The phase portrait is a spiral converging to the origin. □