Final Review

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Summary

For your final, you should...

- Know the material covered in the Midterm Review
- Know how to solve a second order nonhomogeneous linear equation with constant coefficients using the method of undetermined coefficients
- Know how to solve a second order nonhomogeneous linear equation with constant coefficients using the method of variation of parameters
- Know how to solve a homogeneous linear system with real, distinct eigenvalues
- Know how to solve a homogeneous linear system with complex eigenvalues
- Know how to solve a homogeneous linear system with real, repeated eigenvalues
- Know how to find a fundamental matrix of a homogeneous linear system

If you are not sure if you know how to do any of the above, you should read the appropriate notes and do some practice problems from your homework, textbook, etc.

Undetermined Coefficients

The associated section in your textbook is Section 3.5.

Consider the nonhomogeneous equation $y''+P(x)y+Q(x)y=g(x)$ where $P(x),Q(x)$ are constants. If $g(x)$ is of the type

$$p(x)=a_nx^n+...+a_1x+a_0, \quad p(x)e^{\alpha x}, \quad p(x)e^{\alpha x}\sin(\beta x), \quad p(x)e^{\alpha x}\cos(\beta x),$$

we can use a method called the "method of undetermined coefficients". There are actually two methods using undetermined coefficients, one is a substitution approach and the other is called the annihilator approach. In these notes I will only be discussing the substitution approach.

Here are some example of the form we use for y_p:

$g(x)$	Form of y_p
x^3+x^2+x+2	Ax^3+Bx^2+Cx+E
e^{5x}	Ae^{5x}
$\sin(4x)$	$A\sin(4x)+B\cos(4x)$
x^2e^x	$(Ax^2+Bx+C)e^x$
$5x\sin 2x$	$(Ax+B)\sin 2x+(Cx+E)\cos 2x$
$x^2e^x\sin x$	$(Ax^2+Bx+C)e^x\sin x+(Ex^2+Fx+G)e^x\cos x$

We need to be careful with how we pick y_p, however. If $g(x)$ does not have any functions contained in the solution to the homogeneous equation, then y_p is found similar to the equations above. However, if $g(x)$ does contain functions that are in the general solution, then we must multiply our y_p by x^n where n is the smallest integer that eliminates duplication.

For example, consider the equation $y''-2y'+1=e^x$. The general solution to the homogeneous equation $y''-2y'+1=0$ is $y_c=c_1e^x+c_2xe^x$. Notice, however, that $g(x)=e^x$ which is already in our general solution, so we can't use $y_p=e^x$. If we multiply by x, xe^x is still in our general solution, so we need to multiply by x again. Thus the form we would want to use for $y_p=x^2e^x$.

Use the following steps to solve a system of homogeneous linear first order equations:

1. Put into standard form $y''+P(x)y'+Q(x)=g(x)$.
2. Solve the associated homogeneous equation to get the complementary or general solution y_c (or sometimes denoted y_h).
3. Find the form of y_p using the following cases:
   - I. $g(x)$ contains no function of y_c
   - II. $g(x)$ contains a function of y_c
4. Substitute y_p into your equation and solve for the coefficients.
5. The solution to the differential equation is $y=y_c+y_p$.

Variation of Parameters

The associated section in your textbook is Section 3.6.

Consider the nonhomogeneous equation $y'' + P(x)y + Q(x)y = g(x)$ where $P(x), Q(x)$ are constants. We can use a method called Variation of Parameters to solve this equation. This requires using Cramer's Rule and determinants.

Use the following steps to solve a system of homogeneous linear first order equations:

1. Put into standard form $y'' + P(x)y' + Q(x) = g(x)$.
2. Solve the associated homogeneous equation to get the complementary or general solution $y_c = c_1y_1 + c_2y_2$ (or sometimes denoted y_h).
3. Find W, W_1, and W_2 where
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$
4. Find u_1, u_2 where
$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$
5. The particular solution is $y_p = u_1y_1 + u_2y_2$.
6. The solution to the differential equation is $y = y_c + y_p$.

Homogeneous Linear Systems

The associated sections in your textbook are Section 7.5, 7.6, 7.8.

A homogeneous linear first order system will be of the form $\mathbf{x}' = A\mathbf{x}$. For these notes we will assume A is a 2×2 matrix. You have seen 3×3 cases in your homework.

We have the following cases:

I. A has distinct eigenvalues

II. A has repeated eigenvalues

III. A has complex eigenvalues

Use the following steps to solve for Case I (real and distinct eigenvalues):

1. Solve for the eigenvalues λ_1, λ_2 of A by setting $\det(\mathbf{A} - \mathbf{I}\lambda) = 0$.
2. Solve for the corresponding eigenvectors $\mathbf{v_1}, \mathbf{v_2}$ of your eigenvalues.
3. The general solution is $\mathbf{x} = c_1\mathbf{v_1}e^{\lambda_1 t} + c_2\mathbf{v_2}e^{\lambda_2 t}$

4. Solve for your constants if given initial values.

Use the following steps to solve for Case II (real and repeated eigenvalues):

1. Solve for the eigenvalue λ of A by setting $\det(\mathbf{A} - \mathbf{I}\lambda) = 0$.
2. Solve for the first eigenvector as usual; that is, solve the equation $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{v_1} = \mathbf{0}$.
3. Solve for the second eigenvector by solving the equation $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{v_2} = \mathbf{v_1}$.
4. The general solution is $\mathbf{x} = c_1\mathbf{v_1}e^{\lambda t} + c_2[\mathbf{v_1}te^{\lambda t} + \mathbf{v_2}e^{\lambda t}]$.
5. Solve for your constants if given initial values.

Use the following steps to solve for Case III (complex eigenvalues):

1. Solve for the eigenvalues λ_1, λ_2 of A by setting $\det(\mathbf{A} - \mathbf{I}\lambda) = 0$. *Note*: λ_1 and λ_2 should be complex conjugates of each other.
2. Solve for the corresponding eigenvectors $\mathbf{v_1}, \mathbf{v_2}$ of your eigenvalues. *Note*: $\mathbf{v_1}$ and $\mathbf{v_2}$ should be complex conjugates of each other.
3. Write $\mathbf{v_1}e^{\lambda_1 t}$ as a sum of real and imaginary parts using Euler's identity; i.e. write in the form
$$\mathbf{v_1}e^{\lambda_1 t} = \text{Re}(\mathbf{v_1}e^{\lambda_1 t}) + i\text{Im}(\mathbf{v_1}e^{\lambda_1 t})$$
Note: $\mathbf{v_2}e^{\lambda_2 t}$ will have similar real and imaginary parts.
4. The general solution is $\mathbf{x} = c_1\text{Re}(\mathbf{v_1}e^{\lambda_1 t}) + c_2\text{Im}(\mathbf{v_1}e^{\lambda_1 t})$.

Note: It is possible that we can have repeated eigenvalues which return multiple eigenvectors. For example, the system
$$\mathbf{x}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{x}$$
has an eigenvalue $\lambda = 1$ with multiplicity 2 with two linearly independent eigenvectors
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$
Its general solution is therefore
$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}.$$
It is when the number of eigenvectors is less than the multiplicity of the eigenvalue that we must find more eigenvectors using the approach in Case II.

For a 3×3 system with eigenvalue λ of multiplicity 3 but only one eigenvector, the general solution is
$$\mathbf{x} = c_1\mathbf{v}_1e^{\lambda t} + c_2(\mathbf{v}_1te^{\lambda t} + \mathbf{v}_2e^{\lambda t}) + c_3\left(\mathbf{v}_1\frac{t^2}{2}e^{\lambda t} + \mathbf{v}_2te^{\lambda t} + \mathbf{v}_3e^{\lambda t}\right)$$
where $(A - \lambda I)\mathbf{v}_1 = \mathbf{0}, (A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1, (A - \lambda I)\mathbf{v}_3 = \mathbf{v}_2$.

Fundamental Matrices

The associated section in your textbook is Section 7.7.

A fundamental matrix is a matrix of your linearly independent solutions. For example, consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}.$$

This system has the solution

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t},$$

therefore the fundamental matrix of this system is

$$\psi = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}.$$