Solving Homogeneous Systems with Constant Coefficients

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A homogeneous system with constant coefficients can be written in the form $\mathbf{x}' = A\mathbf{x}$ where A is a matrix of constants. For example, a 2×2 system would look something like

$$x' = ax + by$$
$$y' = cx + dy$$

which can then be written in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The remainder of these notes will focus on such 2×2 systems.

The first step is to always find the eigenvalues of the matrix A. We have the following cases for the eigenvalues:

- the eigenvalues are real and distinct (no repeats)
- the eigenvalues are complex
- the eigenvalues are real and repeated.

These cases determine which method we should use to find the solution of the system, as outlined below.

Case 1: Real, distinct eigenvalues

See Section 7.5 of the text.

Suppose λ_1, λ_2 are real and distinct eigenvalues of the matrix A. We then find the corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. The solution to the system is then

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}.$$

Summary of steps:

1. Find the eigenvalues λ_1, λ_2 using the equation $\det(A - \lambda I) = 0$.
2. Find the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ using the equation $(A - \lambda I)\mathbf{v} = 0$.
3. The general solution is $\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$.

Case 2: Complex eigenvalues

See Section 7.6 of the text.

Suppose λ_1, λ_2 are complex eigenvalues of the matrix A and that they are complex conjugates of each other (this is always the case when working with real-valued matrices). We then find the corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, which will also be complex conjugates of each other.

The functions $\mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{v}_2 e^{\lambda_2 t}$ will be solutions to the differential equation, just like in Case 1. However, we want to write our solution in terms of real values, not complex values. We use Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

to write $\mathbf{v}_1 e^{\lambda_1 t}$ in terms of its real and imaginary parts; that is,

$$\mathbf{v}_1 e^{\lambda_1 t} = \mathrm{Re}(\mathbf{v}_1 e^{\lambda_1 t}) + i\mathrm{Im}(\mathbf{v}_1 e^{\lambda_1 t}).$$

If we were to write $\mathbf{v}_2 e^{\lambda_2 t}$ in terms of its real and imaginary parts, we will also get the same real and imaginary parts, just differing by a sign.

The solution to the system is then

$$\mathbf{x} = c_1 \mathrm{Re}(\mathbf{v}_1 e^{\lambda_1 t}) + c_2 \mathrm{Im}(\mathbf{v}_1 e^{\lambda_1 t}).$$

Summary of steps:

1. Find the eigenvalues λ_1, λ_2 using the equation $\det(A - \lambda I) = 0$ (they should be complex conjugates of each other).
2. Find the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ using the equation $(A - \lambda I)\mathbf{v} = 0$ (they should be complex conjugates of each other).
3. Write $\mathbf{v}_1 e^{\lambda_1 t}$ in terms of its real and imaginary parts: $\mathbf{v}_1 e^{\lambda_1 t} = \mathrm{Re}(\mathbf{v}_1 e^{\lambda_1 t}) + i\mathrm{Im}(\mathbf{v}_1 e^{\lambda_1 t})$.
4. The general solution is $\mathbf{x} = c_1 \mathrm{Re}(\mathbf{v}_1 e^{\lambda_1 t}) + c_2 \mathrm{Im}(\mathbf{v}_1 e^{\lambda_1 t})$.

Case 3: Real, repeated eigenvalues

See Section 7.8 of the text.

Suppose λ is an eigenvalue of the matrix A with multiplicity 2. We wish to find two eigenvectors for this eigenvalue. However, when solving the equation $(A - \lambda I)\mathbf{v} = 0$, we will only get one eigenvector, call it $\mathbf{v}_1$. To find a second eigenvector, we then solve the equation

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

The solution to the system is then

$$\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda t} + c_2[\mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}].$$

Summary of steps:

1. Find the eigenvalue λusing the equation $\det(A - \lambda I) = 0$.
2. Find the eigenvector $\mathbf{v}_1$ using the equation $(A - \lambda I)\mathbf{v}_1 = 0$.
3. Find the second eigenvector $\mathbf{v}_2$ using the equation $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$.
4. The general solution is $\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda t} + c_2[\mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}]$.

Examples

Will be covered in class!