# Practice Midterm Solutions

Math 4B: Ordinary Differential Equations
Winter 2016
University of California, Santa Barbara

TA: Victoria Kala

**DO NOT LOOK AT THESE SOLUTIONS UNTIL YOU HAVE ATTEMPTED EVERY PROBLEM ON THE PRACTICE MIDTERM**

February 2, 2016

# Practice Midterm Solutions

February 2, 2016

## Answers

This page contains answers only. Detailed solutions are on the following pages.

1. (a) Answers will vary
   (b) Answers will vary
2. See detailed solution
3. (a) $y = \dfrac{-\frac{1}{2}\cos 2x + C}{xe^x}$
   (b) $(0, \infty)$
4. $\sin^{-1}(y) = \frac{1}{2}x^2 + \sin^{-1}\left(\frac{\pi}{6}\right)$
   or $y = \sin\left(\frac{1}{2}x^2 + \sin^{-1}\left(\frac{\pi}{6}\right)\right)$
5. $y + 2\ln|y-1| = x + 5\ln|x-3| + C$
6. $y^2 \sin x - x^3 y - x^2 + y \ln y - y = e$
7. $y = \begin{cases} \dfrac{\frac{1}{2}x^2}{1+x^2}, & 0 \le x < 1 \\ \dfrac{-\frac{1}{2}x^2 + 1}{1+x^2}, & x \ge 1 \end{cases}$
   *Note that this problem is more difficult than it appears
8. (a) $y = e^{-2x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$
   (b) $y = c_1 + c_2 e^x + c_3 x e^x$
   (c) $y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$
   (d) $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$
9. (a) $W = 2e^{6x}$
   (b) $W = cx^2 e^{-x}$
10. (a) $y_2 = x^2 \ln x$
    (b) See detailed solution
    (c) $y = c_1 x^2 + c_2 x^2 \ln x$

## Detailed Solutions

1. Give an example of a third (3rd) order
   (a) nonlinear differential equation
   (b) linear differential equation

   *Proof.* Recall a third order linear differential equation is given by

   $$a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

   where $a_3, a_2, a_1, a_0$ are a functions of $x$, not of $y$. One way to make a nonlinear differential equation is to have functions of $y$ in front of the derivative terms. Some examples are below:

   (a) $\sqrt{xy}y''' - e^x y' = x^2$ is a nonlinear differential equation because of the term $\sqrt{xy}y'''$.
   (b) $y''' - 2xy'' + \frac{1}{\ln x}y' - \sqrt{e^x + 4}y = 0$ is a linear differential equation because all of the functions in front of the derivative terms are functions of $x$ alone.

   □

2. Verify $P = \dfrac{ce^t}{1+ce^t}$ is a solution to the differential equation $\dfrac{dP}{dt} = P(1-P)$.

   *Proof.* We need to substitute $P$ into the equation above and show that the left hand side equations the right hand side.

   Left hand side: (use the quotient rule or product rule for the derivative)

   $$\frac{dP}{dt} = \frac{(1+ce^t)(ce^t)' - ce^t(1+ce^t)'}{(1+ce^t)^2} = \frac{(1+ce^t)(ce^t) - ce^t(ce^t)}{(1+ce^t)^2} = \frac{ce^t}{(1+ce^t)^2}$$

   Right hand side:

   $$P(1-P) = \frac{ce^t}{1+ce^t}\left(1 - \frac{ce^t}{1+ce^t}\right) = \frac{ce^t}{1+ce^t}\left(\frac{1+ce^t}{1+ce^t} - \frac{ce^t}{1+ce^t}\right) = \frac{ce^t}{1+ce^t}\cdot\frac{1}{1+ce^t} = \frac{ce^t}{(1+ce^t)^2}$$

   The left hand side equals the right hand side, hence $P = \dfrac{ce^t}{1+ce^t}$ is a solution to the differential equation $\dfrac{dP}{dt} = P(1-P)$. □

3. (a) Solve the following first order linear equation:

   $$xy' + (1+x)y = e^{-x}\sin 2x$$

   *Solution.* Write the equation in standard form by dividing the equation on both sides by $x$:

   $$y' + \left(\frac{1+x}{x}\right)y = \frac{e^{-x}\sin 2x}{x}$$

Now we wish to find an integrating factor. From the equation above, we see that

$$P(x) = \frac{1+x}{x} = \frac{1}{x} + 1$$

(remember that $P(x)$ is function in front of the $y$ term in the differential equation). Therefore

$$\mu(x) = \exp\left(\int P(x)dx\right) = \exp\left(\int \left(\frac{1}{x}+1\right)dx\right) = e^{\ln x + x} = e^{\ln x}e^{x} = xe^{x}$$

Multiply the standard form equation above by $\mu(x)$:

$$xe^{x}\left[y' + \left(\frac{1+x}{x}\right)y = \frac{e^{-x}\sin 2x}{x}\right]$$

$$\Rightarrow \quad xe^{x}y' + (1+x)e^{x}y = \sin 2x \tag{1}$$

The left hand side should collapse down to the derivative of the product $\mu(x)y$, that is,

$$\frac{d}{dx}\left(xe^{x}y\right) = \sin 2x. \tag{2}$$

Don't believe it? You can always check:

$$\frac{d}{dx}\left(xe^{x}y\right) = xe^{x}y' + (xe^{x})'y = xe^{x}y' + (x'e^{x} + x(e^{x})')y = xe^{x}y' + (e^{x} + xe^{x})y$$

which is the same as the equation (1).
Now looking back at equation (2), we integrate both sides:

$$xe^{x}y = \int \sin 2x dx \quad \Rightarrow \quad xe^{x}y = -\frac{1}{2}\cos 2x + C$$

Solve for $y$:

$$y = \frac{-\frac{1}{2}\cos 2x + C}{xe^{x}}$$

□

(b) For the differential equation in part (a), determine the largest interval on which the existence and uniqueness theorem for first order linear differential equations guarantees the existence of a unique solution at the initial value $y(4) = 1$.

*Solution.* We must look at the standard form equation:

$$y' + \left(\frac{1+x}{x}\right)y = \frac{e^{-x}\sin 2x}{x}$$

We have that

$$P(x) = \frac{1+x}{x}, \quad f(x) = \frac{e^{-x}\sin 2x}{x}$$

The domain of $P(x)$ is all numbers $x$ such that $x \neq 0$. The domain of $f(x)$ is also all numbers $x$ such that $x \neq 0$. Therefore the domain of the equation is all numbers $x$ such that $x \neq 0$, which can be written as $(-\infty, 0) \cup (0, \infty)$.

Now let's look at the initial value $y(4) = 1$. This says when $x = 4$ we have that $y = 1$. $x = 4$ falls into the interval $(0, \infty)$, hence this is the largest interval on which we are guaranteed a unique solution. □

4. Solve the following initial value problem:

$$\frac{dy}{dx} = x\sqrt{1 - y^2}, \qquad y(0) = \frac{\pi}{6}$$

*Solution.* This is a separable equation. We can separate the variables as follows:

$$\frac{dy}{\sqrt{1-y^2}} = xdx$$

Integrate both sides:

$$\int \frac{dy}{\sqrt{1-y^2}} = \int xdx$$

$$\Rightarrow \quad \sin^{-1}(y) = \frac{1}{2}x^2 + C \tag{3}$$

Here we will plug in the initial value (you can also solve for $y$ then plug in the initial value if you wish). When $x = 0, y = \frac{\pi}{6}$:

$$\sin^{-1}\left(\frac{\pi}{6}\right) = \frac{1}{2}(0)^2 + C \quad \Rightarrow \quad C = \sin^{-1}\left(\frac{\pi}{6}\right)$$

Therefore (3) becomes

$$\sin^{-1}(y) = \frac{1}{2}x^2 + \sin^{-1}\left(\frac{\pi}{6}\right)$$

This is the general solution in implicit form. If we were asked to solve for explicit form, then we would need to solve for $y$:

$$y = \sin\left(\frac{1}{2}x^2 + \sin^{-1}\left(\frac{\pi}{6}\right)\right).$$

□

5. Solve the following separable equation:

$$\frac{dy}{dx} = \frac{xy + 2y - x - 2}{xy - 3y + x - 3}$$

*Solution.* We need to factor the right hand side:

$$\frac{dy}{dx} = \frac{y(x+2) - (x+2)}{y(x-3) + (x-3)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{(x+2)(y-1)}{(x-3)(y+1)}$$

Separate the variables as follows:

$$\frac{y+1}{y-1}dy = \frac{x+2}{x-3}dx$$

Integrate both sides:

$$\int \frac{y+1}{y-1}dy = \int \frac{x+2}{x-3}dx \tag{4}$$

You will need to use substitution or long division to evaluate these integrals. I will do substitution.

Left hand side: Let $u = y - 1$, then $du = dy$ and $y = u + 1$. We substitute these into the integral:

$$\int \frac{y+1}{y-1}dy = \int \frac{u+1+1}{u}du == \int \left(1 + \frac{2}{u}\right) du = u + 2\ln|u| = y - 1 + 2\ln|y-1|$$

Right hand side: Let $v = x - 3$, then $dv = dx$ and $x = v + 3$. We substitute these into the integral:

$$\int \frac{x+2}{x-3}dx = \int \frac{v+3+2}{v}du = \int \left(1 + \frac{5}{v}\right) du = v + 5\ln|v| = x - 3 + 5\ln|x-3|$$

Therefore the equation (4) becomes

$$y - 1 + 2\ln|y-1| = x - 3 + 5\ln|x-3| + C \quad \Rightarrow \quad y + 2\ln|y-1| = x + 5\ln|x-3| + C$$

This would be too difficult to solve for $y$, so we will leave it in implicit form.

□

6. Solve the initial value problem

$$(y^2\cos x - 3x^2y - 2x)dx + (2y\sin x - x^3 + \ln(y))dy = 0, \qquad y(0) = e$$

*Solution.* This equation is of the form $Mdx + Ndy = 0$. Let's check to see if the equation is exact; that is, we need to see if $M_y = N_x$:

$$M = y^2\cos x - 3x^2y - 2x \quad \Rightarrow \quad M_y = 2y\cos x - 3x^2$$
$$N = 2y\sin x - x^3 + \ln(y) \quad \Rightarrow \quad N_x = 2y\cos x - 3x^2$$

$M_y = N_x$ so this equation is exact!

We want to find an $f(x,y)$ such that $\frac{\partial f}{\partial x} = M$, so we will integrate $M$ with respect to $x$:

$$\begin{aligned} f(x,y) &= \int M dx + g(y) \\ &= \int (y^2\cos x - 3x^2y - 2x)dx + g(y) \\ &= y^2\sin x - x^3y - x^2 + g(y) \end{aligned}$$

We want $\frac{df}{dy} = N$, so we will differentiate $f$ with respect to $y$ and set it equal to $N$:

$$\frac{\partial f}{\partial y} = N \quad \Rightarrow \quad \frac{\partial}{\partial y}(y^2 \sin x - x^3 y - x^2 + g(y)) = 2y \sin x - x^3 + \ln(y)$$

$$\Rightarrow \quad 2y \sin x - x^3 + g'(y) = 2y \sin x - x^3 + \ln y$$

Solve for $g'(y)$:

$$g'(y) = \ln y$$

and integrate:

$$g(y) = \int \ln y dy.$$

You will need to use integration by parts to evaluate this integral. Try $u = \ln y, dv = dy$. Then $du = \frac{1}{y}dy, v = y$ and we have

$$g(y) = \int \ln y dy = y \ln y - \int y \cdot \frac{1}{y} dy = y \ln y - \int dy = y \ln y - y$$

Substituting this into our equation above for $f$, we have

$$f(x, y) = y^2 \sin x - x^3 y - x^2 + y \ln y - y.$$

Now set $f(x, y) = C$:

$$y^2 \sin x - x^3 y - x^2 + y \ln y - y = C$$

This is the general solution to the differential equation. Now we need to plug in the initial value $y(0) = e$. When $x = 0, y = e$:

$$e^2 \sin 0 - (0)^3 \cdot e - (0)^2 + e \ln e = C \quad \Rightarrow \quad C = e$$

Therefore the solution is $y^2 \sin x - x^3 y - x^2 + y \ln y - y = e$. □

7. Solve the initial value problem

$$(1 + x^2)\frac{dy}{dx} + 2xy = f(x), \qquad y(0) = 0$$

where $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ -x, & x \geq 1 \end{cases}$. *Hint:* an integrating factor has already been multiplied through.

*Solution.* This is a first order linear differential equation. Because $f(x)$ is a piecewise function, we really have two differential equations. When $0 \leq x < 1$, we have

$$(1 + x^2)\frac{dy}{dx} + 2xy = x, \quad y(0) = 0$$

and when $x \geq 1$

$$(1 + x^2)\frac{dy}{dx} + 2xy = -x, \quad y(1) = ?$$

Notice the initial condition on the interval $x \geq 1$ is different than the initial condition on the interval $0 \leq x < 1$. We won't know what this different initial condition is until we solve the differential equation on the interval $0 \leq x < 1$.

Let's look at the differential equation on $0 \leq x < 1$:

$$(1+x^2)\frac{dy}{dx} + 2xy = x, \quad y(0) = 0.$$

Using the hint that an integration factor has been multiplied through, the left hand side collapses down to a product rule:

$$[(1+x^2)y]' = x$$

Don't believe it? Check it! Then integrate both sides:

$$(1+x^2)y = \int x dx \quad \Rightarrow \quad (1+x^2)y = \frac{1}{2}x^2 + C$$

Solve for $y$:

$$y = \frac{\frac{1}{2}x^2 + C}{1+x^2}$$

Then plug in the initial condition $y(0) = 0$. When $x = 0, y = 0$:

$$0 = \frac{\frac{1}{2}(0)^2 + C}{1+(0)^2} \quad \Rightarrow \quad C = 0$$

Therefore the solution on $0 \leq x < 1$ is given by

$$y = \frac{\frac{1}{2}x^2}{1+x^2}.$$

Before we solve the differential equation on the interval $x \geq 1$, we want to find out the initial condition at $x = 1$. To do this we plug in $x = 1$ into the solution above:

$$y(1) = \frac{\frac{1}{2}(1)^2}{1+(1)^2} = \frac{1}{4}.$$

Therefore the IVP on $x \geq 1$ is

$$(1+x^2)\frac{dy}{dx} + 2xy = -x, \quad y(1) = \frac{1}{4}.$$

Using the hint that an integration factor has been multiplied through, the left hand side collapses down to a product rule:

$$[(1+x^2)y]' = -x$$

Don't believe it? Check it! Then integrate both sides:

$$(1+x^2)y = \int -x dx \quad \Rightarrow \quad (1+x^2)y = -\frac{1}{2}x^2 + C$$

Solve for $y$:

$$y = \frac{-\frac{1}{2}x^2 + C}{1 + x^2}$$

Then plug in the initial condition $y(1) = \frac{1}{4}$. When $x = 1, y = \frac{1}{4}$:

$$\frac{1}{4} = \frac{-\frac{1}{2}(1)^2 + C}{1 + (1)^2} \quad \Rightarrow \quad \frac{1}{2} = -\frac{1}{2} + C \quad \Rightarrow \quad C = 1$$

Therefore the solution on $x \geq 1$ is given by

$$y = \frac{-\frac{1}{2}x^2 + 1}{1 + x^2}.$$

We can write the solution as a the piecewise function:

$$y = \begin{cases} \dfrac{\frac{1}{2}x^2}{1 + x^2}, & 0 \leq x < 1 \\ \dfrac{-\frac{1}{2}x^2 + 1}{1 + x^2}, & x \geq 1 \end{cases}.$$

□

8. Find the general solution of the following homogeneous higher-order differential equations:

(a) $y'' + 4y' + 7y = 0$ (use $x$ as the independent variable)

*Solution.* The characteristic equation is $r^2 + 4r + 7 = 0$. Using the quadratic equation, we see the roots are

$$r = \frac{-4 \pm \sqrt{4^2 - 4(1)(7)}}{2} \quad \Rightarrow \quad r = \frac{-4 \pm i2\sqrt{3}}{2} = -2 \pm i\sqrt{3}.$$

We have two complex roots, hence the solution to the differential equation is

$$y = e^{-2x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

□

(b) $y^{(3)} + 2y'' + y' = 0$ (use $x$ as the independent variable)

*Solution.* The characteristic equation is $r^3 + 2r^2 + r = 0$ which factors as

$$r(r^2 + 2r + 1) = 0 \quad \Rightarrow \quad r(r + 1)^2 = 0.$$

Therefore the roots are $r = 0$ multiplicity 1, $r = -1$ multiplicity 2. The solution to the differential equation is therefore

$$y = c_1 e^{0x} + c_2 e^{1x} + c3xe^{1x} \quad \text{or} \quad y = c_1 + c_2 e^x + c_3 x e^x.$$

□

(c) $2\frac{d^2u}{dt^2} - 5\frac{du}{dt} - 3u = 0$

*Solution.* The characteristic equation is $2r^2 - 5r - 3 = 0$, which factors as

$$(2r+1)(r-3) = 0.$$

The roots are $r = -\frac{1}{2}, r = 3$ each with multiplicity 1. The solution to the differential equation is therefore

$$y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}.$$

□

(d) $\frac{d^4r}{ds^4} - r = 0$

*Solution.* The characteristic equation is $m^4 - 1 = 0$ (we cannot use $r$ as our characteristic equation variable since it is being used in the differential equation). This factors as

$$(m^2-1)(m^2+1) = 0 \quad \Rightarrow \quad (m-1)(m+1)(m-i)(m+i) = 0.$$

The roots are $m = 1, -1, \pm i$. The solution to the differential equation is

$$y = c_1 e^x + c_2 e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x) \quad \Rightarrow \quad y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x.$$

□

9. (a) Calculate $W(y_1, y_2, y_3)$ where $y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x}$.

*Solution.*

$$\begin{aligned}
W(e^x, e^{2x}, e^{3x}) &= \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ (e^x)' & (e^{2x})' & (e^{3x})' \\ (e^x)'' & (e^{2x})'' & (e^{3x})'' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} \\
&= e^x \begin{vmatrix} 2e^{2x} & 3e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix} - e^x \begin{vmatrix} e^{2x} & e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix} + e^x \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} \\
&= e^x(18e^{5x} - 12e^{5x}) - e^x(9e^{5x} - 4e^{5x}) + e^x(3e^{5x} - 2e^{3x}) \\
&= e^x(6e^{5x} - 5e^{5x} + e^{5x}) \\
&= 2e^{6x}
\end{aligned}$$

□

(b) Suppose $y_1, y_2$ are solutions to the equation $x^2y'' + (2x - x^2)y' + y = 0$. Find $W(y_1, y_2)$.

*Proof.* We need to write the differential equation in standard form by dividing through by $x^2$:

$$y'' + \frac{2x - x^2}{x^2}y' + \frac{1}{x^2}y = 0.$$

We see that $P(x) = \frac{2x-x^2}{x^2} = \frac{2}{x} - 1$. Therefore, by using Abel's Theorem,

$$\begin{aligned} W(y_1, y_2) &= c\exp\left(\int P(x)dx\right) = c\exp\left(\int \left(\frac{2}{x} - 1\right) dx\right) = c\exp\left(2\ln x - x\right) \\ &= ce^{2\ln x}e^{-x} = ce^{\ln x^2}e^{-x} = cx^2e^{-x} \end{aligned}$$

□

10. (a) The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Use the method of reduction of order to find a second solution $y_2$ to the differential equation on the interval $(0, \infty)$.

*Proof.* We need to write the differential equation in standard form by dividing through by $x^2$:

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0 \tag{5}$$

Let $y_2 = vy_1$ where $v$ is a function of $x$. Since $y_1 = x^2$, then $y_2 = x^2v$. We wish to plug this into the equation above, so we need to find $y_2', y_2''$ using the product rule:

$$\begin{aligned} y_2 &= x^2v \\ y_2' &= (x^2)'v + x^2v' = 2xv + x^2v' \\ y_2'' &= (2xv)' + (x^2v')' = (2x)'v + 2xv' + (x^2)'v' + x^2v'' = 2v + 2xv' + 2xv' + x^2v'' = 2v + 4xv' + x^2v'' \end{aligned}$$

Substitute these into the standard form equation (5) above:

$$2v + 4xv' + x^2v'' - \frac{3}{x}(2xv + x^2v') + \frac{4}{x^2}(x^2v) = 0$$

$$\Rightarrow \quad 2v + 4xv' + x^2v'' - 6xv - 3xv' + 4v = 0$$

$$\Rightarrow \quad x^2v'' + xv' = 0 \tag{6}$$

Now let $w = v'$, then $w = v''$ and (6) becomes

$$x^2w' + xw = 0 \quad \Rightarrow \quad x^2\frac{dw}{dx} = -xw.$$

This is a separable equation. Separate the variables:

$$\frac{dw}{w} = -\frac{1}{x}dx$$

Integrate both sides:

$$\int \frac{dw}{w} = -\int \frac{1}{x}dx \quad \Rightarrow \quad \ln w = -\ln x \quad \Rightarrow \quad \ln w = \ln\left(\frac{1}{x}\right)$$

Solve for $w$:

$$w = \frac{1}{x}.$$

But, $w = v'$, and so we have

$$v' = \frac{1}{x} \quad \Rightarrow \quad v = \int \frac{1}{x} dx = \ln x$$

Therefore

$$y_2 = x^2 v \quad \Rightarrow \quad y_2 = x^2 \ln x.$$

□

(b) Show that $y_1$ and $y_2$ form a fundamental set of solutions to the differential equation.

*Solution.* We need to show that $W(y_1, y_2) \neq 0$ in the interval $(0, \infty)$.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} x^2 & x^2 \ln x \\ (x^2)' & (x^2 \ln x)' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} \\ &= x^2(2x \ln x + x) - 2x(x^2 \ln x) \\ &= x^3 \end{aligned}$$

$W = x^3$ which is nonzero since we are in the interval $(0, \infty)$. □

(c) Write the general solution of the differential equation using $y_1$ and $y_2$.

*Solution.* The general solution to the differential equation is $y = c_1 y_1 + c_2 y_2$, which is

$$y = c_1 x^2 + c_2 x^2 \ln x.$$

□