Practice Final Solutions

Math 4B: Ordinary Differential Equations
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DO NOT LOOK AT THESE SOLUTIONS UNTIL YOU HAVE ATTEMPTED EVERY PROBLEM ON THE PRACTICE FINAL

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Answers

This page contains answers only. Detailed solutions are on the following pages.

1. $\ln|y-1| - \ln|y+1| = \ln|x-1| - \ln|x+1|$ or $y = x$
2. $y = \dfrac{x^2+C}{(x+1)e^x}$
3. (a) $k = \frac{9}{2}$
   (b) $3x^2y^3 + x\cos y = C$
4. $\frac{1}{2}\ln(x^2+4) + \frac{1}{2}y^2 = C$ or $\frac{1}{2}x^2e^{y^2} + 2e^{y^2} = C$
5. See detailed solution
6. $y_2 = -t^{1/2}$
7. (a) $y = 5e^t + 5te^t$
   (b) $y = \frac{5}{36} - \frac{5}{36}e^{-6t} + \frac{1}{6}te^{-6t}$
   (c) $y = 2\cos 4t - \frac{1}{2}\sin 4t$
8. (a) $y_p = At^2 + Bt + C$
   (b) Answers will vary
   (c) $y_p = At^2 + Bt + C + Ee^{-2t}$
   (d) $y_p = Ae^{3t}\sin 4t + Be^{3t}\cos 4t$
   (e) Answers will vary
   (f) $y_p = A + (Bt^2 + Ct + E)e^{-t}$
   (g) $y_p = A\cos 2t + B\sin 2t$
   (h) Answers will vary
   (i) $y_p = At + B + (Ct+E)\sin t + (Ft+G)\cos t$
9. $y = c_1 + c_2e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$
10. $y = 2e^{-2t} + 9te^{-2t} + \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t}$
11. $y = c_1\cos t + c_2\sin t - \cos t\ln|\sec t + \tan t|$
12. $y = c_1\cos(\ln t) + c_2\sin(\ln t) + \ln|\cos(\ln t)|(\cos(\ln t)) + \ln|t|(\sin(\ln t))$
13. $\mathbf{x} = 3\begin{pmatrix}1\\1\end{pmatrix}e^{t/2} + 2\begin{pmatrix}0\\1\end{pmatrix}e^{-t/2}$
14. $\mathbf{x} = \begin{pmatrix}-1\\6\end{pmatrix}e^{4t} + \begin{pmatrix}26\\13\end{pmatrix}te^{4t}$
15. $\mathbf{x} = e^{5t}\begin{pmatrix}-2\cos 2t - 5\sin 2t\\ 8\cos 2t - 9\sin 2t\end{pmatrix}$
16. $\mathbf{x} = c_1\begin{pmatrix}1\\1\\0\end{pmatrix}e^{-t} + c_2\begin{pmatrix}-1\\0\\1\end{pmatrix}e^{-t} + c_3\begin{pmatrix}1\\-1\\1\end{pmatrix}e^{5t}$
17. $\mathbf{x} = c_1\begin{pmatrix}1\\0\\0\end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix}1\\0\\0\end{pmatrix}te^{2t} + \begin{pmatrix}0\\1\\0\end{pmatrix}e^{2t}\right] + c_3\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\frac{t^2}{2}e^{2t} + \begin{pmatrix}0\\1\\0\end{pmatrix}te^{2t} + \begin{pmatrix}0\\-\frac{6}{5}\\\frac{1}{5}\end{pmatrix}e^{2t}\right]$
18. $\mathbf{x} = c_1\begin{pmatrix}0\\1\\0\end{pmatrix}e^{6t} + c_2e^{4t}\begin{pmatrix}\cos 2t\\0\\-2\sin 2t\end{pmatrix} + c_3e^{4t}\begin{pmatrix}\sin 2t\\0\\2\cos 2t\end{pmatrix}$

Detailed Solutions

1. Solve the given initial-value problem:

$$\frac{dy}{dx} = \frac{y^2-1}{x^2-1}, \quad y(2) = 2$$

Solution. This is a separable equation. Rewrite with y's on one side and x's on the other:

$$\frac{dy}{y^2-1} = \frac{dx}{x^2-1}.$$

We must use partial fraction decomposition to integrate both sides. Start by factoring the denominators:

$$\frac{dy}{(y-1)(y+1)} = \frac{dx}{(x-1)(x+1)}$$

The partial fraction decomposition of the left hand side is

$$\frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}.$$

To find A, B, clear the denominators by multiplying both sides by $(y-1)(y+1)$ and combining like terms:

$$\begin{aligned} 1 &= A(y+1) + B(y-1) \\ 1 &= Ay + A + By - B \\ 1 &= (A+B)y + A - B \end{aligned}$$

Set like terms equal to each other to get the following system:

$$\begin{cases} A+B &= 0 \\ A-B &= 1 \end{cases}$$

which has the solution $A = \frac{1}{2}, B = -\frac{1}{2}$. Therefore the partial fraction decomposition of the left hand side is

$$\frac{1}{(y-1)(y+1)} = \frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}.$$

The right hand side also has the same partial fraction decomposition:

$$\frac{1}{(x-1)(x+1)} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}.$$

We now integrate both sides:

$$\begin{gathered} \int \left(\frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1} \right) dy = \int \left(\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1} \right) dx \\ \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \\ \ln|y-1| - \ln|y+1| = \ln|x-1| - \ln|x+1| + C \end{gathered}$$

We will now plug in the initial value. When $x = 2, y = 2$:

$$\ln|2-1| - \ln|2+1| = \ln|2-1| - \ln|2+1| + C \quad \Rightarrow \quad C = 0.$$

Therefore the implicit solution is

$$\ln|y-1| - \ln|y+1| = \ln|x-1| - \ln|x+1|.$$

Since the problem didn't give instructions of whether to solve for the solution implicitly or explicitly, you can leave it in implicit form. If you wish to solve for it explicitly, combine the log terms:

$$\ln\left|\frac{y-1}{y+1}\right| = \ln\left|\frac{x-1}{x+1}\right|$$

Take the exponential of both sides and then solve for y

$$\frac{y-1}{y+1} = \frac{x-1}{x+1} \quad \Rightarrow \quad y-1 = \frac{x-1}{x+1}(y+1) \quad \Rightarrow \quad y - \frac{x-1}{x+1}y = \frac{x-1}{x+1}+1$$

$$\Rightarrow \quad y\left(1-\frac{x-1}{x+1}\right) = \frac{x-1}{x+1}+1 \quad \Rightarrow \quad y = \frac{\frac{x-1}{x+1}+1}{1-\frac{x-1}{x+1}} \quad \Rightarrow \quad y = \frac{x-1+x+1}{x+1-(x-1)}$$

$$y = x$$

□

2. Solve the following differential equation:

$$(x+1)\frac{dy}{dx} = -(x+2)y + 2xe^{-x}$$

Solution. This is a first order linear equation. Rewrite in standard form:

$$(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x} \quad \Rightarrow \quad \frac{dy}{dx} + \frac{x+2}{x+1}y = \frac{2xe^{-x}}{x+1}$$

We have that

$$P(x) = \frac{x+2}{x+1} = 1 + \frac{1}{x+1}$$

(use long division or write $x+2 = x+1+1$), therefore our integrating factor is

$$\mu = \exp\left(\int P(x)dx\right) = \exp\left(\int\left(1+\frac{1}{x+1}\right)dx\right) = \exp(x + \ln|x+1|) = e^x e^{\ln|x+1|} = (x+1)e^x.$$

Multiply μ by the standard form equation:

$$(x+1)e^x\left(\frac{dy}{dx} + \frac{x+2}{x+1}y\right) = (x+1)e^x\left(\frac{2xe^{-x}}{x+1}\right)$$

$$(x+1)e^x\frac{dy}{dx} + (x+2)e^x y = 2x \quad \Rightarrow \quad (x+1)e^x\frac{dy}{dx} + [(x+1)e^x]'y = 2x$$

$$([(x+1)e^x]y)' = 2x$$

Integrate both sides:

$$(x+1)e^x y = \int 2x dx \quad \Rightarrow \quad (x+1)e^x y = x^2 + C$$

Solve for y:

$$y = \frac{x^2+C}{(x+1)e^x}$$

□

3. Consider the differential equation $(6xy^3 + \cos y)dx + (2kx^2y^2 - x\sin y)dy = 0$.

   (a) Find the value of k so that the given differential equation is exact.

   (b) Solve the equation using your value of k from part (a).

Solution. (a) We need to find the value of k such that $M_y = N_x$.

$$M = 6xy^3 + \cos y \qquad N = 2kx^2y^2 - x\sin y$$

$$M_y = 18xy^2 - \sin y \quad N_x = 4kxy^2 - \sin y$$

We need $4k = 18$, or $k = \frac{9}{2}$.

(b) Substitute $k = \frac{9}{2}$ into the equation above:

$$(6xy^3 + \cos y)dx + (9x^2y^2 - x\sin y)dy = 0.$$

Let's check that our equation is indeed exact.

$$M = 6xy^3 + \cos y \quad N = 9x^2y^2 - x\sin y$$

$$M_y = 18xy^2 - \sin y \quad N_x = 18xy^2 - \sin y$$

$M_y = N_x$ so our equation is exact. Recall that we need to find an f such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$. First we integrate M with respect to x:

$$\begin{aligned} f(x,y) &= \int M dx + g(y) = \int (6xy^3 + \cos y)dx + g(y) \\ &= 3x^2y^3 + x\cos y + g(y) \end{aligned}$$

Then we take the derivative of f with respect to y and set it equal to N:

$$\frac{\partial}{\partial y}(3x^2y^3 + x\cos y + g(y)) = 9x^2y^2 - x\sin y$$

$$9x^2y^2 - x\sin y + g'(y) = 9x^2y^2 - x\sin y \quad \Rightarrow \quad g'(y) = 0$$

Therefore $g(y) = const.$. Plug this into f above and set $f(x,y) = C$:

$$3x^2y^3 + x\cos y = C.$$

□

4. Show that the equation

$$(x^2y + 4y)dy + xdx = 0$$

is not exact, then find the appropriate integrating factor to make the equation exact. Solve the initial value problem given $y(4) = 0$.

Hint: Put your equation in the correct form before you begin.

Solution. Rewrite the equation in the form $Mdx + Ndy = 0$:

$$xdx + (x^2y + 4y)dy = 0.$$

We need to show that the equation is not exact, so we need to show that $M_y \neq N_x$.

$$M = x \quad N = x^2y + 4y$$

$$M_y = 0 \quad N_x = 2xy$$

Clearly $M_y \neq N_x$.

Now we will try and find an integrating factor:

- Check if $\frac{M_y - N_x}{N}$ is a function of x alone:

$$\frac{M_y - N_x}{N} = \frac{-2xy}{x^2y + 4y} = \frac{-2xy}{(x^2+4)y} = -\frac{2x}{x^2+4}$$

  is a function of x alone, so this is a possibility.
- Check if $\frac{N_x - M_y}{M}$ is a function of y alone:

$$\frac{N_x - M_y}{M} = \frac{2xy}{x} = 2y$$

  is a function of y alone, so this is also a possibility.
- Check if $\frac{N_x - M_y}{xM - yN}$ is a function of xy:

$$\frac{N_x - M_y}{xM - yN} = \frac{2xy}{x^2 - x^2y^2 - 4y^2}$$

  is not a function of xy alone, so we will not use this method.

So we see that we could use either the first two to find an integrating factor. We will show both.

If we use the first equation:

$$\mu = \exp\left(\int -\frac{2x}{x^2+4}dx\right) = \exp\left(-\ln(x^2+4)\right) = \frac{1}{x^2+4}$$

Multiply μ by the equation:

$$\frac{x}{x^2+4}dx + ydy = 0$$

Let's check to see if this equation is exact:

$$M = \frac{x}{x^2+4} \quad N = y$$

$$M_y = 0 \quad N_x = 0$$

Clearly $M_y = N_x$, so the equation is indeed exact. We need to find an f such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$. First we integrate M with respect to x (use the substitution $u = x^2 + 4, du = 2xdx$):

$$f(x,y) = \int \frac{x}{x^2+4}dx + g(y) = \frac{1}{2}\ln(x^2+4) + g(y)$$

Then we take the derivative of f with respect to y and set it equal to N:

$$\frac{\partial}{\partial y}\left(\frac{1}{2}\ln(x^2+4) + g(y)\right) = y$$

$$g'(y) = y$$

Integrate $g'(y)$:

$$g(y) = \frac{1}{2}y^2$$

Plug this into f above and set $f(x,y) = C$:

$$\frac{1}{2}\ln(x^2+4) + \frac{1}{2}y^2 = C.$$

If we use the second equation:

$$\mu = \exp\left(\int \frac{N_x - M_y}{M}dy\right) = \exp\left(\int 2ydy\right) = e^{y^2}$$

Multiply μ by the equation:
$$xe^{y^2}dx + (x^2ye^{y^2} + 4ye^{y^2})dy = 0$$

Let's check to see if this equation is exact:

$$M = xe^{y^2} \quad N = x^2ye^{y^2} + 4ye^{y^2}$$

$$M_y = 2xye^{y^2} \quad N_x = 2xye^{y^2}$$

Clearly $M_y = N_x$, so the equation is indeed exact. We need to find an f such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$. First we integrate M with respect to x:

$$\begin{aligned} f(x,y) &= \int M dx + g(y) = \int xe^{y^2} dx + g(y) \\ &= \frac{1}{2}x^2e^{y^2} + g(y) \end{aligned}$$

Then we take the derivative of f with respect to y and set it equal to N:

$$\frac{\partial}{\partial y}\left(\frac{1}{2}x^2e^{y^2} + g(y)\right) = x^2ye^{y^2} + 4ye^{y^2}$$

$$x^2ye^{y^2} + g'(y) = x^2ye^{y^2} + 4ye^{y^2} \quad g'(y) = 4ye^{y^2}.$$

Integrate $g'(y)$ using the substitution $u = y^2, du = 2ydy$:

$$g(y) = \int 4ye^{y^2}dy = \int 2e^u du = 2e^u = 2e^{y^2}$$

Plug this into f above and set $f(x,y) = C$:

$$\frac{1}{2}x^2e^{y^2} + 2e^{y^2} = C.$$

You can verify yourself that these two solutions are equivalent. □

5. Show that $y_1 = x^2$ and $y_2 = x^2 \ln x$ are linearly independent solutions of the homogeneous equation $x^3y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$.

Solution. We need to show two things:

- y_1 and y_2 are solutions to the given equation
- y_1 and y_2 are linearly independent

Let's first verify that $y_1 = x^2$ is a solution to the given differential equation. Find y_1', y_1'', y_1''':

$$\begin{aligned} y_1 &= x^2 \\ y_1' &= 2x \\ y_1'' &= 2 \\ y_1''' &= 0 \end{aligned}$$

Then
$$x^3y''' - 2xy' + 4y = x^2(0) - 2x(2x) + 4(x^2) = 0,$$

as desired.

Now let's verify that $y_2 = x^2 \ln x$ is a solution to the given differential equation. Find y_2', y_2'', y_2''':

$$\begin{aligned} y_2 &= x^2 \ln x \\ y_2' &= 2x \ln x + x \\ y_2'' &= 2 \ln x + 2 + 1 = 2 \ln x + 3 \\ y_2''' &= \frac{2}{x} \end{aligned}$$

Then

$$\begin{aligned} x^3 y''' - 2xy' + 4y &= x^3 \left(\frac{2}{x}\right) - 2x(2x \ln x + x) + 4x^2 \ln x \\ &= 2x^2 - 4x^2 \ln x - 2x^2 + 4x^2 \ln x \\ &= 0, \end{aligned}$$

as desired.

Now let's show that the solutions are linearly independent. We need to show that $W(y_1, y_2) \neq 0$:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = x^2(2x \ln x + x) - 2x^2 \ln x = x^3 \neq 0,$$

since we are on the domain $(0, \infty)$. Therefore the solutions are linearly independent. □

6. Use reduction of order to find a second solution y_2 to the differential equation

$$4t^2 y'' + y = 0$$

given that $y_1 = t^{1/2} \ln t$ is a solution.

Solution. Let $y_2 = vy_1 = v(t^{1/2} \ln t)$. We need to find y_2', y_2'':

$$\begin{aligned} y_2' &= v'(t^{1/2} \ln t) + v(t^{1/2} \ln t))' = v'(t^{1/2} \ln t) + v\left(\frac{1}{2} t^{-1/2} \ln t + t^{-1/2}\right) \\ &= v'(t^{1/2} \ln t) + v\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) \\ y_2'' &= v''(t^{1/2} \ln t) + v'(t^{1/2} \ln t)' + v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) + v\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right)' \\ &= v''(t^{1/2} \ln t) + 2v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) + v\left(-\frac{1}{2} t^{-3/2}\left(\frac{1}{2} \ln t + 1\right) + t^{-1/2} \cdot \frac{1}{2t}\right) \\ &= v''(t^{1/2} \ln t) + 2v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) + v\left(-\frac{1}{4} t^{-3/2} \ln t\right) \end{aligned}$$

Now plug these into the original equation:

$$\begin{aligned} 4t^2 y'' + y &= 4t^2\left(v''(t^{1/2} \ln t) + 2v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) + v\left(-\frac{1}{4} t^{-3/2} \ln t\right)\right) + v(t^{1/2} \ln t) \\ &= 4v'' t^{5/2} \ln t + 8t^2 v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) - t^2 v\left(t^{-3/2} \ln t\right) + v(t^{1/2} \ln t) \\ &= 4v'' t^{5/2} \ln t + 8t^2 v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) \end{aligned}$$

Now set this equal to 0:

$$4v'' t^{5/2} \ln t + 8t^2 v'\left(t^{-1/2}\left(\frac{1}{2} \ln t + 1\right)\right) = 0$$

Let $w = v'$, then $w' = v''$:

$$4w't^{5/2}\ln t + 8t^2 w\left(t^{-1/2}\left(\frac{1}{2}\ln t + 1\right)\right) = 0$$

This is a separable equation. We can rewrite w' as $\frac{dw}{dx}$ and get w's on one side, t's on the other:

$$4\frac{dw}{dt}t^{5/2}\ln t = -8t^2 w\left(t^{-1/2}\left(\frac{1}{2}\ln t + 1\right)\right)$$

$$\frac{dw}{w} = \frac{-2t^2\left(t^{-1/2}\left(\frac{1}{2}\ln t + 1\right)\right)}{t^{5/2}\ln t}$$

Simplify the right hand side:

$$\frac{-t^{3/2}\ln t - 2t^{3/2}}{t^{5/2}\ln t} = -\frac{1}{t} - \frac{2}{t\ln t}$$

Integrate both sides:

$$\int \frac{dw}{w} = \int \left(-\frac{1}{t} - \frac{2}{t\ln t}\right) dt$$

On the right hand side you will need to use the substitution $a = \ln t, da = \frac{1}{t}dt$:

$$\ln|w| = -\ln|t| - 2\ln|\ln t|$$

Now solve for w. But first we need to combine the log terms on the right hand side:

$$\ln|w| = \ln\left|\frac{1}{t}\right| + \ln\left|\frac{1}{(\ln t)^2}\right|$$

$$\ln|w| = \ln\left|\frac{1}{t(\ln t)^2}\right|$$

Take the exponential of both sides:

$$w = \frac{1}{t(\ln t)^2}$$

But, $w = v'$, so

$$v = \int \frac{1}{t(\ln t)^2}dt$$

and using the substitution $a = \ln t, da = \frac{1}{t}dt$,

$$v = \int \frac{1}{a^2}da = -\frac{1}{a} = -\frac{1}{\ln t.}$$

Therefore

$$y_2 = v_2 t^{1/2}\ln t = -\frac{1}{\ln t}\cdot t^{1/2}\ln t = -t^{1/2}.$$

□

7. Solve the following initial-value problems:

   (a) $y'' - 2y' + y = 0, y(0) = 5, y'(0) = 10$

   (b) $y''' + 12y'' + 36y' = 0, y(0) = 0, y'(0) = 1, y''(0) = -7$

   (c) $y'' + 16y = 0, y(0) = 2, y'(0) = -2$

Solution. (a) $y'' - 2y' + y = 0$ has the characteristic equation

$$r^2 - 2r + 1 = 0 \quad \Rightarrow \quad (r-1)^2 = 0 \quad r = 1 \text{ mult. } 2$$

Therefore the general solution is $y = c_1e^t + c_2te^t$.We need to plug in the given initial conditions. When $t = 0, y = 5$:

$$5 = c_1 + c_2 \cdot 0 \quad \Rightarrow \quad c_1 = 5.$$

Therefore $y = 5e^t + c_2te^t$. Find y':

$$y' = 5e^t + c_2(e^t + te^t)$$

When $t = 0, y = 10$:

$$10 = 5 + c_2 \quad \Rightarrow \quad c_2 = 5.$$

Therefore $y = 5e^t + 5te^t$.

(b) $y''' + 12y'' + 36y' = 0$ has the characteristic equation

$$r^3 + 12r^2 + 36r = 0 \quad \Rightarrow \quad r(r^2 + 12r + 36) = 0 \quad \Rightarrow \quad r(r+6)^2 = 0 \quad \Rightarrow \quad r = 0, r = -6 \text{ mult. } 2$$

Therefore the general solution is $y = c_1e^{0t} + c_2e^{-6t} + c_3te^{-6t}$, or $y = c_1 + c_2e^{-6t} + c_3te^{-6t}$. Find y' and y'':

$$\begin{aligned} y' &= -6c_2e^{-6t} + c_3(e^{-6t} - 6te^{-6t}) \\ y'' &= 36c_2e^{-6t} + c_3(-6e^{-6t} - 6e^{-6t} + 36te^{-6t}) \end{aligned}$$

Plug in the initial conditions. When $t = 0, y = 0, y' = 1, y'' = -7$:

$$\begin{aligned} 0 &= c_1 + c_2 \\ 1 &= -6c_2 + c_3 \\ -7 &= 36c_2 - 12c_3 \end{aligned}$$

The solution to this system is $c_1 = \frac{5}{36}, c_2 = -\frac{5}{36}, c_3 = \frac{1}{6}$. Therefore $y = \frac{5}{36} - \frac{5}{36}e^{-6t} + \frac{1}{6}te^{-6t}$.

(c) $y'' + 16y = 0$ has the characteristic equation

$$r^2 + 16 = 0 \quad \Rightarrow \quad r = \pm 4i$$

Therefore the general solution is $y = c_1 \cos 4t + c_2 \sin 4t$. Find y':

$$y' = -4c_1 \sin 4t + 4c_2 \cos 4t.$$

When $t = 0, y = 2, y' = -2$:

$$\begin{aligned} 2 &= c_1 \\ -2 &= 4c_2 \end{aligned}$$

which has the solution $c_1 = 2, c_2 = -\frac{1}{2}$. Therefore $y = 2\cos 4t - \frac{1}{2}\sin 4t$.

□

8. Suppose you are solving the equation $y'' + P(t)y' + Q(t)y = g(t)$, where $P(t)$ and $Q(t)$ are constants, using the method of undetermined coefficients. Complete the table below. Assume $g(t)$ has no function in common with the homogeneous solution y_h.

$g(t)$	Form of y_p
$3t^2 - 2$	(a)
(b)	Ae^{5t}
$6t^2 + 2 - 12e^{-2t}$	(c)
$e^{3t} \sin 4t$	(d)
(e)	$(At + B)e^{-3t}$
$1 - t^2 e^{-t}$	(f)
$3 \cos 2t$	(g)
(h)	$At + B$
$4t(1 + 3 \sin t)$	(i)

List your solutions for (a) - (i) below:

Solution. (a) $g(t)$ is a second degree polynomial, so we must guess a second degree polynomial: $y_p = At^2 + Bt + C$

(b) y_p is an exponential function, so g must have also been an exponential function: $g = 100e^{5t}$ (answers will vary)

(c) $g(t)$ is a second degree polynomial added to an exponential function, so we must guess a second degree polynomial added to an exponential function: $y_p = At^2 + Bt + C + Ee^{-2t}$.

(d) $g(t)$ is an exponential function multiplied by a sine function, so we must also guess an exponential function multiplied by a sine function (we will also have to include a cosine function since sine and cosine come in pairs): $y_p = Ae^{3t} \sin 4t + Be^{3t} \cos 4t$

(e) y_p is a first degree polynomial multiplied by an exponential function, so g must have been a first degree polynomial multiplied by an exponential function: $g = 18te^{-3t}$ (answers will vary)

(f) $g(t)$ is a constant term added to a second degree polynomial multiplied by an exponential function, so we guess the same thing: $y_p = A + (Bt^2 + Ct + E)e^{-t}$

(g) $g(t)$ is a cosine function, so we must also guess a cosine function (we will also have to include a sine function since sine and cosine come in pairs): $y_p = A \cos 2t + B \sin 2t$

(h) y_p is a first degree polynomial, so g must have also been a first degree polynomial: $g = -t + 3$ (answers will vary)

(i) Distribute first:

$$g = 4t + 12t \sin t$$

g is a first degree polynomial added to a first degree polynomial multiplied by a sine function, so we must also guess the same thing (we will also have to include a cosine function since sine and cosine come in pairs): $y_p = At + B + (Ct + E) \sin t + (Ft + G) \cos t$

□

9. Solve the given differential equation using the method of undetermined coefficients.

$$y'' + 2y' = 3 + 4 \sin 2t.$$

Solution. We first find the homogeneous solution:

$$y'' + 2y' = 0$$

has the characteristic equation

$$r^2 + 2r = 0 \quad \Rightarrow \quad r(r+2) = 0 \quad r = 0, r = -2$$

therefore the homogeneous solution is $y_h = c_1 + c_2 e^{-2t}$.

Now we guess a particular solution. $g(t) = 3 + 4\sin 2t$, which is a constant term added to a sine term. We must also guess the same thing, but we will need to include a cosine term as well since sine and cosine come in pairs:

$$y_p = A + B\sin 2t + C\cos 2t.$$

Now we compare with y_h to check if there are any solutions in common. Notice that y_p has a constant term in common with y_h (c_1), so we must multiply our constant A by t ("bump it up") so there is no more overlap:

$$y_p = At + B\sin 2t + C\cos 2t.$$

We compare again with y_h, and we see there are no more solutions in common, so this is our final guess.

Find y_p', y_p'':

$$\begin{aligned} y_p' &= A + 2B\cos 2t - 2C\sin 2t \\ y_p'' &= -4B\sin 2t - 4C\cos 2t \end{aligned}$$

Substitute these values into our equation above:

$$\begin{aligned} y'' + 2y' &= -4B\sin 2t - 4C\cos 2t + 2(A + 2B\cos 2t - 2C\sin 2t) \\ &= -4B\sin 2t - 4C\cos 2t + 2A + 4B\cos 2t - 4C\sin 2t \end{aligned}$$

Set this equal to $g(t)$:

$$-4B\sin 2t - 4C\cos 2t + 2A + 4B\cos 2t - 4C\sin 2t = 3 + 4\sin 2t$$

Set like terms equal to each other:

$$\begin{aligned} 2A &= 3 \\ -4B - 4C &= 4 \\ -4C + 4B &= 0 \end{aligned}$$

The solution to this system is $A = \frac{3}{2}, B = -\frac{1}{2}, C = -\frac{1}{2}$, therefore the particular solution is $y_p = \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$. The general solution is $y = y_h + y_p$:

$$y = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t.$$

□

10. Solve the given initial-value problem.

$$y'' + 4y' + 4y = (3+t)e^{-2t}, \quad y(0) = 2, y'(0) = 5$$

Solution. We first find the homogeneous solution:

$$y'' + 4y' + 4y = 0$$

has the characteristic equation

$$r^2 + 4r + 4 = 0 \quad \Rightarrow \quad (r+2)^2 = 0 \quad \Rightarrow \quad r = -2 \text{ mult. } 2$$

therefore the homogeneous solution is $y_h = c_1 e^{-2t} + c_2 t e^{-2t}$.

We can use either the method of undetermined coefficients or variation of parameters to solve this equation. We will do both.

Undetermined coefficients: We need to guess our particular solution. $g(t) = (3+t)e^{-2t}$ is a first degree polynomial multiplied by an exponential term. We must also guess the same thing:

$$y_p = (At + B)e^{-2t}$$

Now we compare with y_h to check if there are any solutions in common. Notice that y_p has both e^{-2t} and te^{-2t} in common with y_h, so we must multiply y_p by t ("bump it up"):

$$y_p = t(At + B)e^{-2t}$$

We compare again with y_h, and we see that y_p has te^{-2t} in common with y_h, so we must multiply y_p again by t (" bump it up"):

$$y_p = t^2(At + B)e^{-2t}$$

We compare again with y_h, and we see there are no more solutions in common, so this is our final guess.

Find y_p', y_p'':

$$\begin{aligned}
y_p &= (At^3 + Bt^2)e^{-2t} \\
y_p' &= (3At^2 + 2Bt)e^{-2t} - 2(At^3 + Bt^2)e^{-2t} \\
y_p'' &= (6At + 2B)e^{-2t} - 2(3At^2 + 2Bt)e^{-2t} - 2(3At^2 + 2Bt)e^{-2t} + 4(At^3 + Bt^2)e^{-2t}
\end{aligned}$$

Substitute these values into our equation above:

$$\begin{aligned}
y'' + 4y' + 4y &= (6At + 2B)e^{-2t} - 4(3At^2 + 2Bt)e^{-2t} + 4(At^3 + Bt^2)e^{-2t} \\
&\qquad + 4\left((3At^2 + 2Bt)e^{-2t} - 2(At^3 + Bt^2)e^{-2t}\right) + 4(At^3 + Bt^2)e^{-2t} \\
&= (6At + 2B)e^{-2t} - 4(3At^2 + 2Bt)e^{-2t} + 4(At^3 + Bt^2)e^{-2t} + 4(3At^2 + 2Bt)e^{-2t} \\
&\qquad - 8(At^3 + Bt^2)e^{-2t} + 4(At^3 + Bt^2)e^{-2t} \\
&= (6At + 2B)e^{-2t}
\end{aligned}$$

Set this equal to $g(t)$:

$$(6At + 2B)e^{-2t} = (3 + t)e^{-2t}$$

Set like terms equal to each other:

$$\begin{aligned}
6A &= 1 \\
2B &= 3
\end{aligned}$$

The solution to this system is $A = \frac{1}{6}, B = \frac{3}{2}$, therefore the particular solution is $y_p = \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t}$. The general solution is $y = y_h + y_p$:

$$y = c_1 e^{-2t} + c_2 t e^{-2t} + \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t}.$$

Variation of Parameters: Let $y_1 = e^{-2t}$, $y_2 = te^{-2t}$ (these come from y_h above). Then

$$\begin{aligned}
W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-2t}(e^{-2t} - 2te^{-2t}) - (-2e^{-2t})(te^{-2t}) = e^{-4t} \\
W_1 &= \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & te^{-2t} \\ (3+t)e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = -(3+t)e^{-2t}(te^{-2t}) = -(3+t)te^{-4t} \\
W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} e^{-2t} & 0 \\ -2e^{-2t} & (3+t)e^{-2t} \end{vmatrix} = e^{-2t}(3+t)e^{-2t} = (3+t)e^{-4t}
\end{aligned}$$

Find u_1, u_2:

$$u_1' = \frac{W_1}{W} = \frac{-(3+t)te^{-4t}}{e^{-4t}} = -3t - t^2 \quad \Rightarrow \quad u_1 = \int (-3t - t^2)dt = -\frac{3}{2}t^2 - \frac{1}{3}t^3$$
$$u_2' = \frac{W_2}{W} = \frac{(3+t)e^{-4t}}{e^{-4t}} = 3 + t \quad \Rightarrow \quad u_2 = \int (3+t)dt = 3t + \frac{1}{2}t^2$$

The particular solution is $y_p = u_1 y_1 + u_2 y_2$:

$$\begin{aligned} y_p &= \left(-\frac{3}{2}t^2 - \frac{1}{3}t^3\right)e^{-2t} + \left(3t + \frac{1}{2}t^2\right)te^{-2t} \\ &= -\frac{3}{2}t^2e^{-2t} - \frac{1}{3}t^3e^{-2t} + 3te^{-2t} + \frac{1}{2}t^3e^{-2t} \\ &= \frac{3}{2}t^2e^{-2t} + \frac{1}{6}t^3e^{-2t} \end{aligned}$$

The general solution is $y = y_h + y_p$:

$$y = c_1e^{-2t} + c_2te^{-2t} + \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t}.$$

We aren't finished! We need to find c_1, c_2 given $y(0) = 2, y'(0) = 5$. Find y':

$$\begin{aligned} y &= c_1e^{-2t} + c_2te^{-2t} + \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t} \\ y' &= -2c_1e^{-2t} + c_2(e^{-2t} - 2te^{-2t}) + \left(\frac{1}{2}t^2 + 3t\right)e^{-2t} - 2\left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t} \end{aligned}$$

When $t = 0, y = 2, y' = 5$:

$$\begin{aligned} 2 &= c_1 \\ 5 &= -2c_1 + c_2 \end{aligned}$$

The solution to this system is $c_1 = 2, c_2 = 9$, therefore the general solution is

$$y = 2e^{-2t} + 9te^{-2t} + \left(\frac{1}{6}t^3 + \frac{3}{2}t^2\right)e^{-2t}.$$

□

11. Find the general solution of the given differential equation using variation of parameters.

$$y'' + y = \tan t$$

Solution. We first find the homogeneous solution:

$$y'' + y = 0$$

has the characteristic equation

$$r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i$$

therefore the homogeneous solution is $y = c_1 \cos t + c_2 \sin t$.
Let $y_1 = \cos t, y_2 = \sin t$. Then

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \\ W_1 &= \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin t \\ \tan t & \cos t \end{vmatrix} = -\sin t \tan t = -\frac{\sin^2 t}{\cos t} \\ W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} \cos t & 0 \\ -\sin t & \tan t \end{vmatrix} = \cos t \tan t = \sin t \end{aligned}$$

Find u_1 and u_2:

$$u_1' = \frac{W_1}{W} = -\frac{\sin^2 t}{\cos t}$$

$$\Rightarrow \quad u_1 = -\int \frac{\sin^2 t}{\cos t} dt = -\int \frac{1-\cos^2 t}{\cos t} dt = -\int (\sec t - \cos t) dt = -\ln|\sec t + \tan t| + \sin t$$

$$u_2' = \frac{W_2}{W} = \sin t$$

$$\Rightarrow \quad u_2 = \int \sin t dt = -\cos t$$

The particular solution is $y_p = u_1 y_1 + u_2 y_2$:

$$y_p = (-\ln|\sec t + \tan t| + \sin t)\cos t + \sin t \cos t = -\cos t \ln|\sec t + \tan t|$$

The general solution is $y = y_h + y_p$:

$$y = c_1 \cos t + c_2 \sin t - \cos t \ln|\sec t + \tan t|.$$

□

12. $y_1 = \cos(\ln t), y_2 = \sin(\ln t)$ are independent solutions of the equation

$$t^2 y'' + ty' + y = \sec(\ln t).$$

Find the general solution of the equation.

Solution. Use variation of parameters to solve. Write the equation in standard form:

$$y'' + \frac{1}{t}y' + \frac{1}{t^2}y = \frac{\sec(\ln t)}{t^2}$$

We are given that the homogeneous solution is

$$y_h = c_1 \cos(\ln t) + c_2 \sin(\ln t).$$

Let $y_1 = \cos(\ln t), y_2 = \sin(\ln t)$. Then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\ln t) & \sin(\ln t) \\ -\frac{\sin(\ln t)}{t} & \frac{\cos(\ln t)}{t} \end{vmatrix} = \frac{\cos^2(\ln t)}{t} + \frac{\sin^2(\ln t)}{t} = \frac{1}{t}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin(\ln t) \\ \frac{\sec(\ln t)}{t^2} & \frac{\cos(\ln t)}{t} \end{vmatrix} = -\frac{\sin(\ln t)\sec(\ln t)}{t^2} = -\frac{\tan(\ln t)}{t^2}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} \cos(\ln t) & 0 \\ -\frac{\sin(\ln t)}{t} & \frac{\sec(\ln t)}{t^2} \end{vmatrix} = \frac{1}{t^2}$$

Find u_1 and u_2:

$$u_1' = \frac{W_1}{W} = \frac{-\frac{\tan(\ln t)}{t^2}}{\frac{1}{t}} = -\frac{\tan(\ln t)}{t}$$

$$\Rightarrow \quad u_1 = -\int \frac{\tan(\ln t)}{t} dt = -\int \tan a da = -\int \frac{\sin a}{\cos a} da = \int \frac{1}{b} db = \ln|b| = \ln|\cos(\ln t)|$$

$$u_2' = \frac{W_2}{W} = \frac{\frac{1}{t^2}}{\frac{1}{t}} = \frac{1}{t}$$

$$\Rightarrow \quad u_2 = \int \frac{1}{t} dt = \ln|t|$$

For u_1 we used the substitutions $a = \ln t, da = \frac{1}{t}dt$, and $b = \cos a, db = -\sin a da$. The particular solution is $y_p = u_1 y_1 + u_2 y_2$:

$$y = \ln|\cos(\ln t)|(\cos(\ln t)) + \ln|t|(\sin(\ln t)).$$

The general solution is $y = y_h + y_p$:

$$y = c_1 \cos(\ln t) + c_2 \sin(\ln t) + \ln|\cos(\ln t)|(\cos(\ln t)) + \ln|t|(\sin(\ln t)).$$

□

13. Solve the given initial-value problem:

$$\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Solution. We first find the eigenvalues. Our matrix is a lower triangular matrix, therefore the eigenvalues are the diagonal entries: $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$. You can also calculate these the usual way:

$$\begin{vmatrix} \frac{1}{2} - \lambda & 0 \\ 1 & -\frac{1}{2} - \lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda\right)\left(-\frac{1}{2} - \lambda\right) = 0 \quad \Rightarrow \quad \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}.$$

Now use the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvectors. When $\lambda_1 = \frac{1}{2}$:

$$\left(\begin{array}{cc|c} \frac{1}{2} - \frac{1}{2} & 0 & 0 \\ 1 & -\frac{1}{2} - \frac{1}{2} & 0 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right)$$

The second row tells us that $x_1 = x_2$, x_2 is free. Therefore the first eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. When $\lambda_2 = -\frac{1}{2}$:

$$\left(\begin{array}{cc|c} \frac{1}{2} + \frac{1}{2} & 0 & 0 \\ 1 & -\frac{1}{2} + \frac{1}{2} & 0 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right)$$

The first row tells us that $x_1 = 0$, x_2 is free. Therefore the second eigenvector is $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Since λ_1, λ_2 are real and distinct, then the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2}.$$

Now plug in the initial condition. When $t = 0$, $\mathbf{x} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$:

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which yields the system

$$\begin{aligned} 3 &= c_1 \\ 5 &= c_1 + c_2 \end{aligned}$$

The solution to this system is $c_1 = 3, c_2 = 2$. Therefore the solution is

$$\mathbf{x} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2}.$$

□

14. Solve the given initial-value problem:

$$\mathbf{x}' = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

Solution. We first find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda)+4 = 12-8\lambda+\lambda^2+4 = \lambda^2-8\lambda+16 = (\lambda-4)^2 = 0 \quad \Rightarrow \quad \lambda = 4 \text{ mult. } 2$$

Now use the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvectors. When $\lambda = 4$:

$$\left(\begin{array}{cc|c} 2-4 & 4 & 0 \\ -1 & 6-4 & 0 \end{array} \right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array} \right)$$

The second equation tells us that $x_1 = 2x_2$, x_2 is free. Therefore we only get one eigenvector, and it is $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Since we only have one eigenvector, we will have to "bump up" our solution (i.e. there will be a te^{4t} term in our solution). We need to find a second eigenvector, so we now solve the equation $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$:

$$\left(\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array} \right)$$

The second equation tells us that $x_1 = 2x_2 - 1$, x_2 is free. If we let $x_2 = 0$ then the second eigenvector is $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Since $\lambda = 4$ is a real repeated root and we only found one eigenvector in the beginning, then the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + c_2 \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{4t} \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{4t} \right)$$

Now plug in the initial condition. When $t = 0$, $\mathbf{x} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$:

$$\begin{pmatrix} -1 \\ 6 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

which yields the system

$$\begin{aligned} -1 &= 2c_1 - c_2 \\ 6 &= c_1 \end{aligned}$$

The solution to this system is $c_1 = 6, c_2 = 13$. Therefore the solution is

$$\begin{aligned} \mathbf{x} &= 6 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + 13 \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{4t} \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{4t} \right) \\ &= 6 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + 13 \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{4t} + 13 \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{4t} \\ &= \begin{pmatrix} -1 \\ 6 \end{pmatrix} e^{4t} + \begin{pmatrix} 26 \\ 13 \end{pmatrix} te^{4t}. \end{aligned}$$

□

15. Solve the given initial-value problem:

$$\mathbf{x}' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

Solution. We first find the eigenvalues:

$$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = (6-\lambda)(4-\lambda)+5 = 24-10\lambda+\lambda^2+5 = \lambda^2-10\lambda+29 = 0$$

Use the quadratic formula to solve for λ:

$$\lambda = \frac{10 \pm \sqrt{100-4(1)(29)}}{2} = \frac{10 \pm \sqrt{-16}}{2} = \frac{10 \pm 4i}{2} = 5 \pm 2i$$

Therefore $\lambda_1 = 5+2i, \lambda_2 = 5-2i$.

Now use the equation $(A-\lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvectors. When $\lambda = 5+2i$:

$$\left(\begin{array}{cc|c} 6-(5+2i) & -1 & 0 \\ 5 & 4-(5+2i) & 0 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right)$$

The first row says that $(1-2i)x_1 = x_2$. If we choose $x_1 = 1$, then the first eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$.

The second eigenvector will then be $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}$. The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t} + c_2 \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(5-2i)t},$$

however, we wish to write our answer as a real solution. Here we will expand both $\mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{v}_2 e^{\lambda_2 t}$ and see why it is enough to only expand $\mathbf{v}_1 e^{\lambda_1 t}$ to get our solution.

$$\begin{aligned}
\begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t} &= \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{5t} e^{i2t} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{5t}(\cos 2t + i \sin 2t) \\
&= e^{5t} \begin{pmatrix} \cos 2t + i \sin 2t \\ \cos 2t + i \sin 2t - 2i \cos 2t + 2 \sin 2t \end{pmatrix} \\
&= e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} + i e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix} \\
\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(5-2i)t} &= \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{5t} e^{-i2t} = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{5t}(\cos(-2t) + i \sin(-2t)) = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{5t}(\cos 2t - i \sin 2t) \\
&= e^{5t} \begin{pmatrix} \cos 2t - i \sin 2t \\ \cos 2t - \sin 2t + 2i \cos 2t + 2 \sin 2t \end{pmatrix} \\
&= e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} - i e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix}
\end{aligned}$$

Notice that the expansions of $\mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{v}_2 e^{\lambda_2 t}$ only differ by a sign! If you know the real and imaginary part of one of the expansions, you already know the other. This is why we have only done the expansion of $\mathbf{v}_1 e^{\lambda_1 t}$ in section. We plug these expansions into our general solution:

$$\begin{aligned}
\mathbf{x} &= c_1 \left(e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} + i e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix} \right) + c_2 \left(e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} - i e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix} \right) \\
&= (c_1 + c_2) e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} + i(c_1 - c_2) e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix}
\end{aligned}$$

Relabel the constants to get the general solution

$$\mathbf{x} = c_1 e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix},$$

which is just the real and imaginary parts of $\mathbf{v}_1 e^{\lambda_1 t}$.

Now we must plug in the initial value. When $t = 0, x = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$:

$$\begin{pmatrix} -2 \\ 8 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

which yields the system

$$\begin{aligned} -2 &= c_1 \\ 8 &= c_1 - 2c_2 \end{aligned}$$

This system has the solution $c_1 = -2, c_2 = -5$, therefore the solution is

$$\begin{aligned} \mathbf{x} &= -2e^{5t} \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} - 5e^{5t} \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} \\ &= e^{5t} \begin{pmatrix} -2\cos 2t - 5\sin 2t \\ -2\cos 2t - 4\sin 2t - 5\sin 2t + 10\cos 2t \end{pmatrix} \\ &= e^{5t} \begin{pmatrix} -2\cos 2t - 5\sin 2t \\ 8\cos 2t - 9\sin 2t \end{pmatrix} \end{aligned}$$

□

16. Solve $\mathbf{x}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{x}$.

Solution. We begin by finding the eigenvalues:

$$\begin{aligned} \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} &= (1-\lambda) \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & -2 \\ 2 & 1-\lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & 1-\lambda \\ 2 & -2 \end{vmatrix} \\ &= (1-\lambda)[(1-\lambda)(1-\lambda) - 4] + 2[-2(1-\lambda) + 4] + 2[4 - 2(1-\lambda)] \\ &= (1-\lambda)(\lambda^2 - 2\lambda - 3) + 2(2\lambda + 2) + 2(2\lambda + 2) \\ &= \lambda^2 - 2\lambda - 3 - \lambda^3 + 2\lambda^2 + 3\lambda + 4\lambda + 4 + 4\lambda + 4 \\ &= -\lambda^3 + 3\lambda^2 + 9\lambda + 5 \end{aligned}$$

Let $p(\lambda)$ denote the polynomial above. Since the last term of $p(\lambda)$ is 5 and the first term of $p(\lambda)$ is 1, the possible roots of $p(\lambda)$ are $\pm\frac{1,5}{1} = \pm 1, 5$. We then test each of these numbers until we get $p(\lambda) = 0$. Start with 1:

$$p(1) = -1 + 3 + 9 + 5 \neq 0,$$

so 1 is not a zero of $p(\lambda)$. Move onto -1:

$$p(-1) = 1 + 3 - 9 + 5 = 0,$$

so -1 is a zero of $p(\lambda)$. This implies that $\lambda - (-1) = \lambda + 1$ is a factor of $p(\lambda)$. Use long division to divide $p(\lambda)$ by $\lambda + 1$:

$$\frac{-\lambda^3 + 3\lambda^2 + 9\lambda + 5}{\lambda + 1} = -\lambda^2 + 4\lambda + 5$$

Therefore $p(\lambda) = (\lambda + 1)(-\lambda^2 + 4\lambda + 5)$ and can be factored as

$$p(\lambda) = -(\lambda + 1)^2(\lambda - 5).$$

Therefore the eigenvalues are $\lambda = -1$ mult. 2, $\lambda = 5$.

Now we find the eigenvectors. Start with $\lambda = -1$:

$$\left(\begin{array}{ccc|c} 1-(-1) & -2 & 2 & 0 \\ -2 & 1-(-1) & -2 & 0 \\ 2 & -2 & 1-(-1) & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have two rows of zeros, hence we have two free variables. The first row tells us that $x_1 = x_2 - x_3$. We can write the general solution of the system as

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} x_3$$

Choose $x_2 = 1, x_3 = 0$, and then $x_2 = 0, x_3 = 1$ to get the eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Notice that we have 2 eigenvectors which matches the multiplicity of the eigenvalue $\lambda = -1$. This means two things: we do not need to go find any more eigenvectors, AND we do not need to have a te^{-t} as part of our solution because we found two independent eigenvectors (i.e. we do not need to "bump up" the solution). Now we find the eigenvector for when $\lambda = 5$:

$$\left(\begin{array}{ccc|c} 1-5 & -2 & 2 & 0 \\ -2 & 1-5 & -2 & 0 \\ 2 & -2 & 1-5 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(We skipped a couple of steps.) The first row tells us that $x_1 = x_2 + 2x_3$, and the second row tells us that $x_2 = -x_3$. If we choose $x_3 = 1$, then $x_2 = -1$ and $x_1 = 1$. Therefore the third eigenvector is

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}.$$

□

17. Solve $\mathbf{x}' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$.

Solution. We begin by finding the eigenvalues. Because our matrix is an upper triangular matrix, the eigenvalues are the diagonal entries, hence $\lambda = 2$ mult. 3. We now find the eigenvectors for $\lambda = 2$:

$$\left(\begin{array}{ccc|c} 2-2 & 1 & 6 & 0 \\ 0 & 2-2 & 5 & 0 \\ 0 & 0 & 2-2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The second row tells us that $x_3 = 0$, and the first row tells us that $x_2 = -6x_3$. But this implies that $x_2 = 0$. x_1 is our free variable, so we can choose it to be 1, and we get the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We only found one eigenvector, which is less than the multiplicity of our eigenvalue. This means two things: we must find more eigenvectors, and we must have a te^{2t} in our solution (i.e. we must "bump up" our solution). We find the next eigenvector by solving the equation $(A - I\lambda)\mathbf{v}_2 = \mathbf{v}_1$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The second row tells us $x_3 = 0$, and the first row tells us that $x_2 = -6x_3 + 1$, which means that $x_2 = 1$. x_1 is free, so if we choose it to be 0 then the second eigenvector is

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We need to find another eigenvector, so we solve the equation $(A - I\lambda)\mathbf{v}_3 = \mathbf{v}_2$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The second row tells us that $x_3 = \frac{1}{5}$, the first row tells us that $x_2 = -6x_3 = -\frac{6}{5}$. x_1 is free, so if we choose it to be 0 then the third eigenvector is

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix}.$$

Therefore the solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right] + c_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix} e^{2t} \right].$$

□

18. Solve $\mathbf{x}' = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 6 & 0 \\ -4 & 0 & 4 \end{pmatrix} \mathbf{x}$.

Solution. We begin by finding the eigenvalues:

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 6-\lambda & 0 \\ -4 & 0 & 4-\lambda \end{vmatrix} &= (4-\lambda) \begin{vmatrix} 6-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} - 0 + 1 \begin{vmatrix} 0 & 6-\lambda \\ -4 & 0 \end{vmatrix} \\ &= (4-\lambda)(6-\lambda)(4-\lambda) + 4(6-\lambda) \\ &= (6-\lambda)[(4-\lambda)(4-\lambda) + 4] \\ &= (6-\lambda)(\lambda^2 - 8\lambda + 20) \end{aligned}$$

We use the quadratic formula to find the eigenvalues of $\lambda^2 - 8\lambda + 20$:

$$\lambda = \frac{8 \pm \sqrt{64 - 4(1)(20)}}{2} = \frac{8 \pm \sqrt{-16}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$$

Therefore the eigenvalues are $\lambda_1 = 6, \lambda_2 = 4 + 2i, \lambda_3 = 4 - 2i$. When $\lambda_1 = 6$:

$$\left(\begin{array}{ccc|c} 4-6 & 0 & 1 & 0 \\ 0 & 6-6 & 0 & 0 \\ -4 & 0 & 4-6 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

The last row tells us that $x_3 = 0$, the first row tells us that $2x_1 = x_3$ which implies that $x_1 = 0$. x_2 is free, so if we choose it to be 1 then we have the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

When $\lambda_2 = 4 + 2i$:

$$\left(\begin{array}{ccc|c} 4-(4+2i) & 0 & 1 & 0 \\ 0 & 6-(4+2i) & 0 & 0 \\ -4 & 0 & 4-(4+2i) & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} -2i & 0 & 1 & 0 \\ 0 & 2-2i & 0 & 0 \\ -4 & 0 & -2i & 0 \end{array}\right)$$

The second row tells us that $(2-2i)x_2 = 0$, or that $x_2 = 0$. The first row tells us that $2ix_1 = x_3$. If we choose $x_1 = 1$, then we have the eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2i \end{pmatrix}.$$

This implies that the third eigenvector is

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -2i \end{pmatrix}.$$

Therefore the solution to the system is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2i \end{pmatrix} e^{(4+2i)t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -2i \end{pmatrix} e^{(4-2i)t},$$

however we want to write our solution as a real solution. Therefore we must expand $\mathbf{v}_2 e^{\lambda_2 t}$ (you can also do the same for $\mathbf{v}_3 e^{\lambda_3 t}$ as seen in the solution to Problem 15, but it is enough to just do one expansion):

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \\ 2i \end{pmatrix} e^{4t} e^{i2t} &= \begin{pmatrix} 1 \\ 0 \\ 2i \end{pmatrix} e^{4t}(\cos 2t + i \sin 2t) = e^{4t} \begin{pmatrix} \cos 2t + i \sin 2t \\ 0 \\ 2i \cos 2t - 2 \sin 2t \end{pmatrix} \\ &= e^{4t} \begin{pmatrix} \cos 2t \\ 0 \\ -2 \sin 2t \end{pmatrix} + ie^{4t} \begin{pmatrix} \sin 2t \\ 0 \\ 2 \cos 2t \end{pmatrix}. \end{aligned}$$

Therefore the general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{6t} + c_2 e^{4t} \begin{pmatrix} \cos 2t \\ 0 \\ -2 \sin 2t \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} \sin 2t \\ 0 \\ 2 \cos 2t \end{pmatrix}.$$

□