Quiz 3 Solutions

January 31, 2016

1. Given the differential form $[\ln(\frac{1}{x}) + x]dy + [\frac{cy}{x} + y + 2x]dx = 0$, find the constant c such that the equation is exact.

   Solution. The equation above is of the form $Ndy + Mdx = 0$. This problem will be easier if we rewrite the equation using $\ln(\frac{1}{x}) = -\ln(x)$:
$$[-\ln x + x]dy + [\frac{cy}{x} + y + 2x]dx = 0.$$
   For this equation to be exact we require that $M_y = N_x$:
$$M_y = \frac{\partial}{\partial y}\left(\frac{cy}{x} + y + 2x\right) = \frac{c}{x} + 1$$
$$N_x = \frac{\partial}{\partial x}(-\ln x + x) = -\frac{1}{x} + 1$$
   For these two to be equal we must have $c = -1$. □

2. Using the c determined above, solve this equation, when $y(1) = 1$. Leave your answer as an implicit function of y.

   Solution. In Exercise 1 we found that $c = -1$, so our differential equation is
$$[-\ln x + x]dy + [-\frac{y}{x} + y + 2x]dx = 0.$$
   We know this equation is exact from Exercise 1.
   Integrate M with respect to x:
$$f(x, y) = \int Mdx + g(y) = \int(-\frac{y}{x} + y + 2x)dx + g(y) = -y\ln x + xy + x^2 + g(y).$$
   Take the derivative of f with respect to y and set it equal to N:
$$\frac{\partial}{\partial y}(-y\ln x + xy + x^2 + g(y)) = N \quad \Rightarrow \quad -\ln x + x + g'(y) = -\ln x + x.$$
   Therefore $g'(y) = 0$, and if we integrate this we have that $g(y) = C_1$ for some constant C_1. Therefore
$$f(x, y) = -y\ln x + xy + x^2 + C_1.$$
   We set $f(x, y) = C$, and we get
$$-y\ln x + xy + x^2 + C_1 = C \quad \Rightarrow \quad -y\ln x + xy + x^2 = C.$$
   This is the general solution. Now we need to plug in the initial value $y(1) = 1$. When $x = 1$, we have that $y = 1$:
$$-1\ln 1 + 1 + 1 = C \quad \Rightarrow \quad C = 2.$$
   Hence the solution to the initial value problem is
$$-y\ln x + xy + x^2 = 2.$$
   □