Quiz 5 Solutions

March 5, 2016

1. Find the general solution to $y''(t) - 4'y(t) + 5y(t) = 0$, and the IVP when $y(0) = 0$, and $y'(0) = 2$. Your answer should be entirely real.

   Solution. The characteristic equation to $y'' - 4y' + 5y = 0$ is
   $$r^2 - 4r + 5 = 0.$$
   We cannot factor this polynomial, so we use the quadratic formula to solve for the roots:
   $$r = \frac{4 \pm \sqrt{4^2 - 4(1)(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$
   Therefore the general solution is $y = e^{2t}(c_1 \cos t + c_2 \sin t)$.

   Now we plug in the initial values. When $t = 0, y = 0$, hence
   $$0 = e^0(c_1 \cos 0 + c_2 \sin 0) \quad \Rightarrow \quad c_1 = 0.$$
   Therefore $y = c_2 e^{2t} \sin t$. Using product rule, we find that
   $$y' = c_2(2e^{2t} \sin t + e^{2t} \cos t).$$
   When $t = 0, y' = 2$, hence
   $$2 = c_2(2e^0 \sin 0 + e^0 \cos 0) \quad \Rightarrow \quad c_2 = 2.$$
   Therefore $y = 2e^{2t} \sin t$. □

2. Solve the equation $y''(t) - 4y'(t) + 4y(t) = 0$. Verify both solutions you get work.

   Solution. The characteristic equation to $y'' - 4y + 4 = 0$ is
   $$r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0.$$
   The roots are $r = 2$ with multiplicity 2. Therefore the general solution is $y = c_1 e^{2t} + c_2 t e^{2t}$. (The two solutions are $y_1 = e^{2t}, y_2 = te^{2t}$.)

   To verify the solution, we plug it into the original equation. This means we need to find y' and y'':
   $$\begin{aligned} y &= c_1 e^{2t} + c_2 t e^{2t} \\ y' &= 2c_1 e^{2t} + c_2(e^{2t} + 2te^{2t}) \\ y'' &= 4c_1 e^{2t} + c_2(2e^{2t} + 2e^{2t} + 4te^{2t}) = 4c_1 e^{2t} + c_2(4e^{2t} + 4te^{2t}) \end{aligned}$$

Then

$$
\begin{aligned}
y'' - 4y' + 4y &= 4c_1e^{2t} + c_2(4e^{2t} + 4te^{2t}) - 4[2c_1e^{2t} + c_2(e^{2t} + 2te^{2t})] + 4(c_1e^{2t} + c_2te^{2t}) \\
&= 4c_1e^{2t} + 4c_2e^{2t} + 4c_2te^{2t} - 8c_1e^{2t} - 4c_2e^{2t} - 8c_2te^{2t} + 4c_1e^{2t} + 4c_2te^{2t} \\
&= 0,
\end{aligned}
$$

which is what we expected.

Note: An alternate way would be to plug $y_1 = e^{2t}$ into the equation above, then plug $y_2 = te^{2t}$ into the equation above. □