Quiz 6 Solutions

March 5, 2016

1. Given that -1 and 4 are the roots of the characteristic equation, find the general solution of the ODE
$$y'' - 3y' - 4y = 2\sin t + 4.$$

Solution. We are given that the homogeneous solution is
$$y_h = c_1 e^{-t} + c_2 e^{4t}.$$

We will use the method of undetermined coefficients to solve for the particular and general solutions. (The method of variation of parameters can also be used, but is much more difficult.) We are given that $g(t) = 2\sin t + 4$. So we need to guess something with a sine term and a constant. We will guess
$$y_p = A\sin t + B\cos t + C.$$

We need to plug this into the equation above, so we need to find y' and y'':
$$\begin{aligned} y_p' &= A\cos t - B\sin t \\ y_p'' &= -A\sin t - B\cos t. \end{aligned}$$

Then
$$\begin{aligned} y'' - 3y' - 4y &= -A\sin t - B\cos t - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t + C) \\ &= -A\sin t - B\cos t - 3A\cos t + 3B\sin t - 4A\sin t - 4B\cos t - 4C \\ &= (-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t - 4C \\ &= (-5A + 3B)\sin t + (-3A - 5B)\cos t - 4C. \end{aligned}$$

Now set this equal to $g(t)$:
$$(-5A + 3B)\sin t + (-3A - 5B)\cos t - 4C = 2\sin t + 4.$$

Setting the coefficients of like terms equal to each other, we need to solve the system
$$\begin{aligned} -5A + 3B &= 2 \\ -3A - 5B &= 0 \\ -4C &= 4. \end{aligned}$$

The solution to this system is $A = -\frac{5}{17}, B = \frac{3}{17}, C = -1$. Therefore
$$y_p = -\frac{5}{17}\sin t + \frac{3}{17}\cos t - 1,$$

and the general solution is $y = y_h + y_p$:
$$y = c_1 e^{-t} + c_2 e^{4t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t - 1.$$

□

2. Given that 1 is a double root of the characteristic equation, find the general solution of the ODE $y'' - 2y' + y = te^t$.

Solution. We are given that the homogeneous solution is

$$y_h = c_1 e^t + c_2 te^t.$$

We can use either the method of undetermined coefficients or variation of parameters for the particular and general solutions. We will show both methods here.

Undetermined coefficients: We are given that $g(t) = te^t$, which is a first degree polynomial multiplied by an exponential term. We must also guess something with a first degree polynomial multiplied by an exponential term, so we will guess

$$y_p = (At + B)e^t.$$

However, te^t and e^t are already solutions to y_h, so we need to multiply y_p by t ("bump it up"):

$$y_p = t(At + B)e^t.$$

However, te^t is already a solution to y_h, so we need multiply y_p by t again ("bump it up"):

$$y_p = t^2(At + B)e^t = (At^3 + Bt^2)e^t.$$

We need to plug this into the equation above, so we need to find y' and y'':

$$\begin{aligned} y_p' &= (3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t \\ y_p'' &= (6At + 2B)e^t + (3At^2 + 2Bt)e^t + (3At^2 + 2B)e^t + (At^3 + Bt^2)e^t \\ &= (6At + 2B)e^t + 2(3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t. \end{aligned}$$

Then

$$\begin{aligned} y'' - 2y' + y &= (6At + 2B)e^t + 2(3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t - 2[(3At^2 + 2Bt)e^t \\ &\qquad + (At^3 + Bt^2)e^t] + (At^3 + Bt^2)e^t \\ &= (6At + 2B)e^t + 2(3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t - 2(3At^2 + 2Bt)e^t \\ &\qquad - 2(At^3 + Bt^2)e^t + (At^3 + Bt^2)e^t \\ &= (6At + 2B)e^t. \end{aligned}$$

Now set this equal to $g(t)$:

$$(6At + 2B)e^t = te^t.$$

Setting the coefficients of like terms equal to each other, we need to solve the system

$$\begin{aligned} 6A &= 1 \\ B &= 0. \end{aligned}$$

The solution to this system is $A = 1/6, B = 0$. Therefore

$$y_p = \frac{1}{6}t^3e^t,$$

and the general solution is $y = y_h + y_p$:

$$y = c_1e^t + c_2te^t + \frac{1}{6}t^3e^t.$$

Variation of parameters: From the homogeneous solution $y_h = c_1e^t + c_2te^t$ we know that $y_1 = e^t, y_2 = te^t$. Then

$$\begin{aligned}
W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^t(e^t + te^t) - te^{2t} = e^{2t} \\
W_1 &= \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & te^t \\ te^t & e^t + te^t \end{vmatrix} = -t^2e^{2t} \\
W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} e^t & 0 \\ e^t & te^t \end{vmatrix} = te^{2t}.
\end{aligned}$$

Now we find u_1, u_2:

$$\begin{aligned}
u_1' &= \frac{W_1}{W} = \frac{-t^2e^{2t}}{e^{2t}} = -t^2 \quad \Rightarrow \quad u_1 = \int -t^2 dt = -\frac{1}{3}t^3 \\
u_2' &= \frac{W_2}{W} = \frac{te^{2t}}{e^{2t}} = t \quad \Rightarrow \quad u_2 = \int t dt = \frac{1}{2}t^2.
\end{aligned}$$

Therefore $y_h = u_1y_1 + u_2y_2$:

$$y_h = -\frac{1}{3}t^3e^t + \frac{1}{2}t^2 \cdot te^t \quad \Rightarrow \quad y_h = \frac{1}{6}t^3e^t.$$

Thus the general solution is $y = y_h + y_p$:

$$y = c_1e^t + c_2te^t + \frac{1}{6}t^3e^t.$$

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