Quiz 7 Solutions

March 5, 2016

1. $(1-t)y''(t) + ty'(t) - y(t) = 2(t-1)^2e^{-t}, y_1 = e^t, y_2 = t, 0 < t < 1$. Find $W(y_1(t), y_2(t))$.

   Solution.
   $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & t \\ e^t & 1 \end{vmatrix} = e^t - te^t = (1-t)e^t.$$

   □

2. Use variation of parameters to solve the above equation.

   Solution. First, we need to put our equation into standard form to find out what $g(t)$ is:
   $$y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = \frac{2(t-1)^2e^{-t}}{1-t} \quad \Rightarrow \quad y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = -2(t-1)e^{-t}.$$
   Hence $g(t) = -2(t-1)e^{-t}$.

   Now we find W_1, W_2:
   $$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & t \\ -2(t-1)e^{-t} & 1 \end{vmatrix} = 2t(t-1)e^{-t}$$
   $$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} e^t & 0 \\ e^t & -2(t-1)e^{-t} \end{vmatrix} = -2(t-1)$$

   Now find u_1, u_2:
   $$u_1' = \frac{W_1}{W} = \frac{2t(t-1)e^{-t}}{(1-t)e^t} = -2te^{-2t}, \quad u_2 = \frac{W_2}{W} = \frac{-2(t-1)}{(1-t)e^t} = 2e^{-t}$$
   We need to use integration by parts to evaluate u_1 (set $u = -2t, dv = e^{-2t}dt$). Integrating yields
   $$u_1 = \int -2te^{-2t}dt = te^{-2t} + \frac{1}{2}e^{-2t}, \quad u_2 = \int 2e^{-t}dt = -2e^{-t}.$$
   Then the particular solution is $y_p = u_1y_1 + u_2y_2$:
   $$y = \left(te^{-2t} + \frac{1}{2}e^{-2t}\right)e^t - 2te^{-t} = -te^{-t} + \frac{1}{2}e^{-t}.$$
   Thus the general solution is $y = y_h + y_p$:
   $$y = c_1e^t + c_2t - te^{-t} + \frac{1}{2}e^{-t}.$$

   □