# Quiz 8 Solutions

March 5, 2016

1. (a) Find the general solution of the system $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$.

*Solution.* We need to find the eigenvalues of the matrix $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$:

$$\begin{aligned} \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} &= (3-\lambda)(-2-\lambda) + 4 \\ &= -6 - 3\lambda + 2\lambda + \lambda^2 + 4 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1). \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$.

Now we find the eigenvectors. When $\lambda_1 = 2$, we have the augmented matrix

$$\left( \begin{array}{cc|c} 3-\lambda_1 & -2 & 0 \\ 2 & -2-\lambda_1 & 0 \end{array} \right) \Rightarrow \left( \begin{array}{cc|c} 3-2 & -2 & 0 \\ 2 & -2-2 & 0 \end{array} \right) \Rightarrow \left( \begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array} \right)$$

The second row is a multiple of the first row, which is a sign of a dependent system (as desired). First equation tells us that $x_1 - 2x_2 = 0$, or that $x_1 = 2x_2$. $x_2$ is our free variable. Choosing $x_2 = 1$, we have $x_1 = 2$ and we see that our eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(You can also pick other values if you wish; you may instead to choose $x_1 = 1$, for example.) Now, when $\lambda_2 = -1$:

$$\left( \begin{array}{cc|c} 3-\lambda_2 & -2 & 0 \\ 2 & -2-\lambda_2 & 0 \end{array} \right) \Rightarrow \left( \begin{array}{cc|c} 3-(-1) & -2 & 0 \\ 2 & -2-(-1) & 0 \end{array} \right) \Rightarrow \left( \begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array} \right)$$

The first row is a multiple of the second row, which is a sign of a dependent system (as desired). The second equation tells us that $2x_1 - x_2 = 0$, or that $x_2 = 2x_1$. $x_1$ is our free variable. Choosing $x_1 = 1$, we have $x_2 = 2$, and we see that our eigenvector is

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore the solution is $\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}$:

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}.$$

□

(b) Describe the behavior of the solution of the system above as $t \to \infty$.

*Solution.* As $t \to \infty$, $e^{2t} \to \infty$ and $e^{-t} \to 0$, so clearly the $\mathbf{v}_1 e^{\lambda_1 t}$ term will dominate the solution. Therefore $\mathbf{x} \to \infty$ as $t \to \infty$. (You can also split this up into cases as we saw in class, but saying the limit diverges is enough.) □