Quiz 5 Solutions

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1. Find the general solution of the following differential equation using the method of variation of parameters.
$$y'' - 2y' + y = \frac{e^t}{1+t^2}$$

Solution. Find the homogeneous solution. $y'' - 2y' + y = 0$ has the characteristic equation
$$r^2 - 2r + 1 = 0 \quad \Rightarrow \quad (r-1)^2 = 0 \quad \Rightarrow \quad r = 1 \text{ mult } 2$$

Therefore the homogeneous solution is $y = c_1 e^t + c_2 te^t$. Set $y_1 = e^t, y_2 = te^t$.
Our equation is in standard form with $g(t) = \frac{e^t}{1+t^2}$. Find W, W_1, W_2:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$$
$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & te^t \\ \frac{e^t}{1+t^2} & e^t + te^t \end{vmatrix} = -\frac{te^{2t}}{1+t^2}$$
$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{1+t^2} \end{vmatrix} = \frac{e^{2t}}{1+t^2}$$

Find u_1' and u_2' and integrate to find u_1 and u_2:

$$u_1' = \frac{W_1}{W} = -\frac{t}{1+t^2} \quad \Rightarrow \quad u_1 = -\frac{1}{2}\ln(1+t^2)$$
$$u_2' = \frac{W_2}{W} = \frac{1}{1+t^2} \quad \Rightarrow \quad u_2 = \arctan t$$

The particular solution is then $y_p = u_1 y_1 + u_2 y_2$:
$$y_p = -\frac{e^t}{2}\ln(1+t^2) + te^t \arctan t$$

The solution to the equation is therefore $y = y_h + y_p$:
$$y = c_1 e^t + c_2 te^t - \frac{e^t}{2}\ln(1+t^2) + te^t \arctan t.$$

Note: you may also just use the formula for variation of parameters.

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