Examples for Green's Theorem, Cylindrical Coordinates, and Spherical Coordinates

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Green's Theorem Example

Evaluate $\int_C xydx + x^2dy$ where C is the rectangle with vertices $(0,0),(3,0),(3,1),(0,1)$ oriented counter-clockwise.

Solution. This problem may look familiar as it was on the Line Integral "Quiz". To solve this integral as a standard line integral, had to split up our integral along each of the edges of the rectangle (see the solution for more details).

We will now solve this line integral using Green's Theorem. Recall that Green's Theorem is given by

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

For this problem, we have $P = xy, Q = x^2$, and $D = \{(x,y) : 0 \leq x \leq 3, 0 \leq y \leq 1\}$ (D is just the rectangle). Therefore

$$\begin{aligned}\int_C xydy + x^2dy &= \int_0^1 \int_0^3 \left(\frac{\partial(x^2)}{\partial x} - \frac{\partial(xy)}{\partial y}\right) dxdy = \int_0^1 \int_0^3 (2x - x)\, dxdy = \int_0^1 \int_0^3 xdxdy \\ &= \int_0^1 \frac{1}{2}x^2\Big|_{x=0}^3 dy = \int_0^1 \frac{9}{2}dy = \frac{9}{2}y\Big|_{y=0}^1 = \frac{9}{2}\end{aligned}$$

Notice that this is the same solution as we had before. □

Cylindrical Coordinates Example

Find the volume of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$ and below the plane $z = 4$.

Proof. Sketch the graph. The paraboloid intersects with the cylinder when $z = 2$. We really have two volumes that we need to find: the volume of the paraboloid inside the cylinder from $z = 0$ to $z = 2$, and the volume of the cylinder from $z = 2$ to $z = 4$.

For the volume of the paraboloid inside the cylinder from $z = 0$ to $z = 2$, our volume is bounded above by the plane $z = 2$ and bounded below by the paraboloid $z = x^2 + y^2$. We will use polar coordinates, so $x = r\cos\theta, y = r\sin\theta$ and $x^2 + y^2 = r^2$. Therefore our volume is bounded above by the plane $z = 2$ and below by the paraboloid $z = r^2$. The smallest our radius can be is when $r = 0$ (this corresponds to the bottom of the paraboloid), and the largest our radius can be is when $r = 2$

(this is when the paraboloid intersects with the cylinder). We are "going all the way around" our volume so $0 \leq \theta \leq 2\pi$.

For the volume of the cylinder from $z = 2$ to $z = 4$ we are done. We do not need to change any variables in our equations for z. Because we want the volume on the inside of our cylinder, $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. Therefore, the volume we desire is given by

$$
\begin{aligned}
V = \iiint dV &= \int_0^{2\pi} \int_0^2 \int_{r^2}^2 r dz dr d\theta + \int_0^{2\pi} \int_0^2 \int_2^4 r dz dr d\theta \\
&= \int_0^{2\pi} \int_0^2 rz\Big|_{z=r^2}^2 dr d\theta + \int_0^{2\pi} \int_0^2 rz\Big|_{z=2}^4 dr d\theta \\
&= \int_0^{2\pi} \int_0^2 \big(r(2-r^2) + r(4-2)\big)\, dr d\theta = \int_0^{2\pi} \int_0^2 \big(4r - r^3\big)\, dr d\theta \\
&= 2\pi \left(2r^2 - \frac{1}{4}r^4\right)\Big|_{r=0}^2 = 2\pi(8-4) = 8\pi.
\end{aligned}
$$

□

Spherical Coordinates Example

Use a triple integral to find the volume of the sphere of radius 3 in the first octant.

Proof. Sketch the graph. The first octant is when $x, y, z \geq 0$. The radius of our sphere is 3, so we have $0 \leq \rho \leq 3$.

To find θ, we look at the xy-plane. We are only going a quarter of the "way around", therefore $0 \leq \theta \leq \dfrac{\pi}{2}$.

To find the azimuthal angle ϕ, we start at the top of the sphere and move down the sphere until we intersect with the xy-plane, which occurs when $\phi = \frac{\pi}{2}$. Therefore $0 \leq \phi \leq \frac{\pi}{2}$.

Therefore the volume is given by

$$
\begin{aligned}
V = \iiint dV &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi d\rho d\phi d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{3}\rho^3\Big|_0^3 \sin\phi d\phi d\theta \\
&= 9 \int_0^{\pi/2} (-\cos(\phi))\Big|_0^{\pi/2} d\theta = 9 \int_0^{\pi/2} d\theta = \frac{9\pi}{2}.
\end{aligned}
$$

Note: The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, so one-eighth of the volume of the sphere would be $V = \frac{1}{8} \cdot \frac{4}{3}\pi r^3$. Plugging in $r = 3$ would give you the same answer as above. □