Math 6A: Line Integral "Quiz" Solutions

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1. Evaluate $\int_C xydx+x^2dy$ where C is the rectangle with vertices $(0,0),(3,0),(3,1),(0,1)$ oriented counter clockwise.

   Solution. Oriented counter-clockwise means we move around the box counter-clockwise; i.e. we start at $(0,0)$ then move to $(3,0)$, then to (3,1), then to (0,1), and back to (0,0). We will split up our integral along each side of the box and then add all the integrals together at the end. Let C_1 denote the line from $(0,0)$ to $(3,0)$, C_2 denote the line from $(3,0)$ to $(3,1)$, C_3 denote the line from $(3,1)$ to $(0,1)$, and C_4 denote the line from $(0,1)$ to $(0,0)$.

   Along C_1: The starting point is $(0,0)$ and the ending point is $(3,0)$, therefore the parametrization of this line is given by

   $$\mathbf{r}(t) = (1-t)\langle 0,0\rangle + t\langle 3,0\rangle \quad \Rightarrow \quad \mathbf{r}(t) = \langle 3t, 0\rangle, \quad 0 \le t \le 1.$$

   So $x=3t, y=0$ and $dx = 3dt, dy = 0dt$. Plugging these into our line integral, we have

   $$\int_{C_1} xydx + x^2dy = \int_0^1 3t\cdot 0 \cdot 3dt + (3t)^2 \cdot 0dt = 0.$$

   Along C_2: The starting point is $(3,0)$ and the ending point is $(3,1)$, therefore the parametrization of this line is given by

   $$\mathbf{r}(t) = (1-t)\langle 3,0\rangle + t\langle 3,1\rangle \quad \Rightarrow \quad \mathbf{r}(t) = \langle 3, t\rangle, \quad 0 \le t \le 1.$$

   So $x=3, y=t$ and $dx = 0dt, dy = dt$. Plugging these into our line integral, we have

   $$\int_{C_2} xydx + x^2dy = \int_0^1 3\cdot t \cdot 0dt + 3^2dt = \int_0^1 9dt = 9.$$

   Along C_3: The starting point is $(3,1)$ and the ending point is $(0,1)$, therefore the parametrization of this line is given by

   $$\mathbf{r}(t) = (1-t)\langle 3,1\rangle + t\langle 0,1\rangle \quad \Rightarrow \quad \mathbf{r}(t) = \langle 3-3t, 1\rangle, \quad 0 \le t \le 1.$$

   So $x=3-3t, y=1$ and $dx = -3dt, dy = 0dt$. Plugging these into our line integral, we have

   $$\int_{C_3} xydx + x^2dy = \int_0^1 (3-3t)\cdot 1 \cdot (-3)dt + (3-3t)^2\cdot 0dt = \int_0^1 (9t-9)dt = -\frac{9}{2}.$$

   Along C_4: The starting point is $(0,1)$ and the ending point is $(0,0)$, therefore the parametrization of this line is given by

   $$\mathbf{r}(t) = (1-t)\langle 0,1\rangle + t\langle 0,0\rangle \quad \Rightarrow \quad \mathbf{r}(t) = \langle 0, 1-t\rangle, \quad 0 \le t \le 1.$$

   So $x=0, y=1-t$ and $dx = 0dt, dy = -dt$. Plugging these into our line integral, we have

   $$\int_{C_4} xydx + x^2dy = \int_0^1 0\cdot(1-t)\cdot 0dt + 0^2\cdot(-dt) = 0.$$

   Therefore

   $$\int_C xydx + x^2dy = 0 + 9 + \left(-\frac{9}{2}\right) + 0 = \frac{9}{2}.$$

   □

2. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos y\mathbf{j} + xz\mathbf{k}$ and C is given by the parametrization $\mathbf{r}(t) = t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1$.

Solution. $\mathbf{r}'(t) = \langle 3t^2, -2t, 1\rangle$ and $\mathbf{F}(\mathbf{r}(t)) = \langle \sin(t^3), \cos(-t^2), t^3 \cdot t\rangle$, therefore

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle \sin(t^3), \cos(-t^2), t^4\rangle \cdot \langle 3t^2, -2t, 1\rangle dt \\ &= \int_0^1 \left(3t^2 \sin(t^3) - 2t\cos(-t^2) + t^4\right) dt \\ &= -\cos(t^3) + \sin(-t^2) + \frac{1}{5}t^5\Big|_0^1 \\ &= -\cos(1) + \sin(1) + \frac{1}{5} - (-\cos(0) + \sin(0) + 0) \\ &= -\cos(1) + \sin(1) + \frac{6}{5}\end{aligned}$$

Note: a u-substitution was used to evaluate the integral. □

3. A thin wire in the shape of a curve C with linear density $\rho(x, y)$ has **mass**

$$m = \int_C \rho(x, y)ds$$

and **center of mass** $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{m}\int_C x\rho(x, y)ds, \quad \bar{y} = \frac{1}{m}\int_C y\rho(x, y)ds.$$

Find the mass and center of mass of a wire bent in the shape of a semicircle $x^2 + y^2 = 4, x \geq 0$ with linear density $\rho(x, y) = k$ where k is a constant.

Solution. The parametrization of the given semicircle is

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

Notice that $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, hence $||\mathbf{r}'(t)|| = 2$.

The mass of the wire is

$$m = \int_C \rho(x, y)ds = \int_{-\pi/2}^{\pi/2} k||\mathbf{r}'(t)||dt = \int_{-\pi/2}^{\pi/2} 2kdt = 2kt\Big|_{-\pi/2}^{\pi/2} = 2k\pi.$$

The x-coordinate of the center of mass is given by

$$\bar{x} = \frac{1}{m}\int_C x\rho(x, y)ds = \frac{1}{2k\pi}\int_{-\pi/2}^{\pi/2} 2\cos t \cdot k||\mathbf{r}'(t)||dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 2\cos tdt = \frac{2}{\pi}\sin t\Big|_{-\pi/2}^{\pi/2} = \frac{4}{\pi}.$$

The y-coordinate of the center of mass is given by

$$\bar{y} = \frac{1}{m}\int_C y\rho(x, y)ds = \frac{1}{2k\pi}\int_{-\pi/2}^{\pi/2} 2\sin t \cdot k||\mathbf{r}'(t)||dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 2\sin tdt = \frac{2}{\pi}(-\cos t)\Big|_{-\pi/2}^{\pi/2} = 0.$$

Thus the wire has mass $m = 2k\pi$ with center of mass $(\bar{x}, \bar{y}) = (\frac{4}{\pi}, 0)$. □