Math 6A Notes

Written by Victoria Kala
vtkala@math.ucsb.edu
SH 6432u Office Hours: R 12:30 − 1:30pm
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Parametric Equations

If $x = f(t), y = g(t)$, we say that x and y are **parametric equations** of the **parameter** t. The collection of points $(x, y) = (f(t), g(t))$ defines a **parametric curve**. If t is defined on a domain $a \le t \le b$, the initial point of the parametric curve is $(f(a), g(a))$ and the terminal point is $(f(b), g(b))$.

A very common parametrization of a circle with radius r (i.e. $x^2 + y^2 = r^2$) is given by

$$x = r\cos t, \quad y = r\sin t \quad 0 \le t \le 2\pi.$$

The easiest way to parametrize the curve $y = f(x)$ is to let $x = t, y = f(t)$.

The derivative $\dfrac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0.$$

If a curve C is described by the parametric equations $x = f(t), y = g(t), \alpha \le t \le \beta$ where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C (**arc length**) is given by

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Vector Functions

A **vector valued function** is a vector whose components are functions. Consider $\mathbf{r}(t)$ given by

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

$\mathbf{r}(t)$ is vector function from $\mathbb{R}$ (input was t) to $\mathbb{R}^3$ (output was a vector with three components).

The limit of a vector function is the limit of its components (provided they exist), i.e.

$$\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle.$$

The derivative of a vector $\mathbf{r}(t)$ is equal to the derivative of each of its components (provided they exist), i.e.

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

Recall that velocity is given by $\mathbf{r}'$ and acceleration is given by $\mathbf{r}''$. The speed is the magnitude of velocity (see definition of magnitude below).

We can also integrate vector functions:

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt \right) \mathbf{i} + \left(\int_a^b g(t)dt \right) \mathbf{j} + \left(\int_a^b h(t)dt \right) \mathbf{k}.$$

Recall the **magnitude** (or length) of a vector $\mathbf{v} = (v_1, v_2, ..., v_n)$ is given by

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}.$$

The **unit tangent vector** to the curve $\mathbf{r}(t)$ is given by

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}.$$

The arc length of $\mathbf{r}(t)$ from $a \leq t \leq b$ is given by

$$L = \int_a^b ||\mathbf{r}'(t)||dt,$$

which is similar to the arc length we defined earlier for parametric equations. The **arc length function** s is given by

$$s(t) = \int_a^t ||\mathbf{r}'(u)||du,$$

where a is the initial point of $\mathbf{r}(t)$. Sometimes we like to **parametrize a curve with respect to arc length**; to do so, we use the following steps:

1. Find the arc length function s.
2. Use your equation from Step 1 to solve for t in terms of s, i.e. find $t(s)$.
3. Substitute your equation from Step 2 into your function $\mathbf{r}(t)$; i.e. find $\mathbf{r}(t(s))$.

The **curvature** κ of $\mathbf{r}(t)$ is

$$\kappa(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}$$

where $\mathbf{T}$ is the unit tangent vector of $\mathbf{r}(t)$.

Two curves $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ are said to **collide** if they equal each other at the same time, i.e. $\mathbf{r}_1(t) = \mathbf{r}_2(t)$ for some time t. Two curves **intersect** if they equal each other but not at the time time, i.e. $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for $t \neq s$.

Dot Product and Cross Product

The dot product of two vectors $\mathbf{a} = \langle a_1, a_2, ..., a_n \rangle$ and $\mathbf{b} = \langle b_1, b_2, ..., b_n \rangle$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + ... + a_n b_n.$$

Notice that the dot product is scalar (number). Here is also a useful theorem which allows us to calculate the angle between two vectors:

Theorem. *The dot product of two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by*

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos\theta$$

where θ *is the angle between the two vectors.*

Two vectors are perpendicular if and only if their dot product is equal to 0.

The cross product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Notice the cross product is a vector. The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal (normal) to both $\mathbf{a}$ and $\mathbf{b}$. We can also use the cross product to calculate the angle between two vectors:

Theorem 1. *The cross product of two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by*

$$\mathbf{a} \times \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \sin\theta$$

where $0 \leq \theta \leq \pi$ *is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$*.*

Two vectors are parallel if and only if their cross product is equal to the zero vector.

Lines and Planes

A line that contains the point (x_0, y_0, z_0) in the direction $\mathbf{v} = \langle a, b, c \rangle$ is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r} = \langle x, y, z \rangle, \mathbf{r}_0 = \langle x_0, y_0 . z_0 \rangle$. This can be written in parametric form as

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

Recall that two lines $\mathbf{r}_1, \mathbf{r}_2$ intersect if they equal each other at different times, i.e. $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for $t \neq s$. If two lines do not intersect but have the same direction, they are said to be **parallel**; if two lines do not intersect and do not have the same direction, they are said to be **skew**.

A plane that contains the point (x_0, y_0, z_0) with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is given by

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

where $\mathbf{r} = \langle x, y, z \rangle, \mathbf{r}_0 = \langle x_0, y_0 . z_0 \rangle$. This can be written in the form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d.$$

Level Curves and Contour Maps

The **level curves** (or contour curves) of a function f of two variables are the curves with equations $f(x, y) = k$ where k is a constant in the range of f.

Let $P(x, y)$ be a point on a contour map. We can estimate partial derivatives at the point P using our contour map. Using the following strategy:

- f_x at P: Move to the right of your point. If the level curves decrease (i.e. k decreases), then $f_x < 0$; if the level curves increase (i.e. k increases), then $f_x > 0$.
- f_y at P: Move up from your point. If the level curves decrease (i.e. k decreases), then $f_y < 0$; if the level curves increase (i.e. k increases), then $f_y > 0$.

We can also use contour maps to estimate second partial derivatives by looking at the spreading or bunching of the level curves. If the level curves bunch together as you move up or down, then the rate of the partial derivative is increasing. If the level curves spread apart as you move up or down, then the rate of the partial derivative is decreasing. If the level curves stay constant as you move up or down, then rate of the partial derivative is constant. If...

- The partial derivative was increasing at an increasing rate, or decreasing at a decreasing rate, then the second partial derivative is > 0.
- The partial derivative was decreasing at an increasing rate, or increasing at a decreasing rate, then the second partial derivative is < 0.
- If the partial derivative had a constant rate then the second partial derivative is $= 0$.

See the contour example for more information.

Limits of Multivariable Functions

To show a limit of a multivariable function does not exist, we take different paths to show the limits are different on each path. For example, suppose we wanted to show

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist. If we take the path $x = y$, then

$$\lim_{y\to 0} \frac{y^2 - y^2}{y^2 + y^2} = 0.$$

But, if we take the path $x = 0$, then

$$\lim_{y\to 0} \frac{0^2 - y^2}{0^2 + y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1.$$

These two limits are clearly not equal, and so the limit does not exist.

Partial Derivatives

If f is a function of two variables, its partial derivatives are the functions f_x, f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}.$$

When we take the partial derivative with respect to x, y remains constant. When we take the partial derivative with respect to y, x remains constant.

We can take higher partial derivatives as well. The second partial derivatives of a function $f(x,y)$ are:

$$f_{xx} = (f_x)_x, \quad f_{xy} = (f_x)_y, \quad f_{yx} = (f_y)_x, \quad f_{yy} = (f_y)_y.$$

We take the derivative left to right using this notation.

Theorem (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy}, f_{yx} are both continuous on D, then*

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Tangent Planes and Linear Approximations

If f has continuous partial derivatives, an equation of the tangent plane to the surface $z = f(x,y)$ at the point $P(x_0,y_0,z_0)$ is

$$z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

The **linearization** (or **linear approximation**) $L(x,y)$ of a function $f(x,y)$ at the point (x_0,y_0) is the equation

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0),$$

which is just the equation of the tangent plane.

If $z = f(x,y)$, the **total differential** dz is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

dz is approximately equal to Δz, the change in height of our function z.

Directional Derivatives and Gradient

If f is a function of variables x, y, z, the **gradient** of f is

$$\nabla f = \langle f_x, f_y, f_z \rangle.$$

The gradient vector gives the direction of the fastest increase of f. Its magnitude gives us the maximum rate of change.

If f is a differential function, f has **directional derivative** $D_u f(x, y)$ in the direction of a unit vector $\mathbf{u}$ given by

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

We can also use the gradient to write equations of the tangent plane to a surface $f(x, y, z) = k$ at the point $P(x_0, y_0, z_0)$:

$$\nabla f \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

Maximum and Minimum Values

A function of two variables has a **local maximum** at (a, b) with local maximum value $f(a, b)$ if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**. In other words, a local maximum has a function value bigger than everything around it, and a local minimum has a function value smaller than everything around it.

Theorem. *If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.*

We set $f_x = f_y = 0$ to find the critical points (a, b). We then use the second derivative test to classify the critical points.

Theorem (The Second Derivative Test)**.** *Suppose the second partial derivatives of f are continuous on a disk with center (a, b) and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (i.e. (a, b) is a critical point of f). Let*

$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

(i) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(ii) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(iii) If $D < 0$, then $f(a, b)$ is a saddle point.

It may be helpful to remember the formula to D as the following determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2.$$

The matrix above is called the Hessian matrix. You may also use the eigenvalues to classify whether a critical point is a max, min, or saddle.

The Extreme Value Theorem tells us that a function f on a bounded set D can attain its maximum inside or on the boundary of D. To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.

2. Find the extreme values of f on the boundary of D.
3. The largest values from (1) and (2) are the absolute maximum value; the smallest values is the absolute minimum value.

Vector Fields

A **vector field** is a function $\mathbf{F}$ that assigns a point (x, y, z) to a vector $\mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z})$. The gradient ∇f forms a vector field, we call it the **gradient vector field**.

A vector field $\mathbf{F}$ is called a **conservative vector field** if there exists a **potential function** f such that $\nabla f = \mathbf{F}$.

Curl and Divergence

The **curl** of a function $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is defined as

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}.$$

Written out more explicitly:

$$\begin{aligned}
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \mathbf{k} \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\end{aligned}$$

The **divergence** of function $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is defined as

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Notice that the curl of a function is a vector and the divergence of a function is a scalar (just a number).

Line Integrals

If f is a function defined along a smooth curve C defined by $x = x(t), y = y(t), a \leq t \leq b$, then the line integral of f along C is

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt.$$

If C is defined by the parametrization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then this formula can also be written as

$$\int_C f(x,y)ds = \int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)||dt.$$

Another form of a line integral is the following:

$$\int_C P(x,y)dx + Q(x,y)dy = \int_C P(x(t),y(t))x'(t)dt + Q(x(t),y(t))y'(t)dt.$$

The previous definition was for line integrals of scalar functions. We can also evaluate line integrals with vector functions. Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t), a \leq t \leq b$. Then the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt.$$

This also known as "work" or "circulation".

If given a graph of a vector field $\mathbf{F}$ and a path C, we can determine the sign of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$:

- If the path travels against the vector field, then the integral is negative.
- If the path travels along the vector field, then the integral is positive.
- If the path appears to both in a symmetric way, or if the vector field is perpendicular to the path at all times, then the integral is zero.

The Fundamental Theorem of Line Integrals

Theorem (Fundamental Theorem of Line Integrals)**.** *Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function whose gradient vector ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

In other words: the line integral of a conservative vector field is independent of the path. We can use the following theorem to determine when a vector field is conservative.

Theorem. *Let $\mathbf{F}$ be a vector field. The following are conservative:*

(i) $\mathbf{F}$ *is conservative*

(ii) There exists a potential function f such that $\nabla f = \mathbf{F}$

(iii) curl $\mathbf{F} = \mathbf{0}$

(iv) $\mathbf{F}$ *is path independent*

(v) $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for ALL closed paths C*

Note: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, $\mathbf{F}$ is conservative if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

Double Integrals

If $f(x, y) \geq 0$ then the double integral $\iint_R f(x, y)dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x, y)$.

We can estimate double rectangles using Riemann sums:

$$\iint_R f(x, y)dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$

where $\Delta A = \Delta x \Delta y$, m, n are the number of partitions of x, y intervals, respectively, and Δx, Δy are the width of the partitions.

Theorem (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, then*

$$\iint_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy.$$

Fubini's Theorem says that the order of integration doesn't matter when we integrate over rectangles. Fubini's Theorem also applies to other change of coordinates (like polar coordinates).

We can also integrate over regions that are not defined by rectangles. If f is continuous on $D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ then

$$\iint_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx.$$

If f is continuous on $D = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$, then

$$\iint_D f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy.$$

The order of integration may matter depending on the function and region we are integrating over (we saw several examples of this in section).

To find the area over a region D, we can integrate

$$A = \iint_D dA$$

where dA is an area element (could be $dxdy, dydx$, or $rdrd\theta$ as seen in the next section).

Polar Coordinates

We use polar coordinates whenever we integrate over circular regions. We use the parametrization $x = r\cos\theta, y = r\sin\theta$. It follows that $x^2 + y^2 = r^2$.

If f is continuous on a polar region $D = \{(r,\theta) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta.$$

In other words, we substitute the equations above in for x and y and use the area element $dA = r dr d\theta$.

Green's Theorem

We say that a closed curve is **positively oriented** if it is oriented counterclockwise. It is negatively oriented if it is oriented clockwise.

Theorem (Green's Theorem)**.** *Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ with P and Q having continuous partial derivatives on an open region that contains D, then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

In other words, Green's Theorem allows us to change from a complicated line integral over a curve C to a less complicated double integral over the region bounded by C.

Triple Integrals

To find the volume of a solid E, we can integrate

$$V = \iiint_E dV$$

where dV is a volume element.

Triple integrals are very similar to double integrals. Not only do we need to find bounds for x and y, but we also need to find bounds for z. The bounds may have dependence on zero, one, or two variables.

Cylindrical Coordinates

Cylindrical coordinates are an application of polar coordinates into three dimensions. In fact, we use the same substitutions for x and y, and just leave z the same:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

It follows that $r^2 = x^2 + y^2$.

If f is continuous and $E = \{(x,y,z) : (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$ where D is given in polar coordinates by $D = \{(r,\theta) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta)\ rdzdrd\theta.$$

In other words, we substitute the equations above in for x and y and use the volume element $dV = rdrd\theta dz$.

Spherical Coordinates

Spherical coordinates are another application of polar coordinates in three dimensions, but now we have three parameters instead of two. Our parameters are ρ (r in polar coordinates), θ (same as in polar coordinates), and ϕ, known as the azimuthal angle. The azimuthal angle is measured from the positive z-axis to the negative z-axis.

The substitutions for x, y, and z are as follows:

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi.$$

It follows that $\rho^2 = x^2 + y^2 + z^2$.

If f is continuous and $E = \{(\rho,\theta,\phi) : a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$, then

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi\ d\rho\ d\theta\ d\phi.$$

In other words, we substitute the equations above in for x, y, z and use the volume element $dV = \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$.

So when do we use cylindrical coordinates versus spherical coordinates? As the names of these methods imply, use cylindrical coordinates for cylinders, and spherical coordinates for spheres. But these certainly aren't the only shapes we integrate over. My rule of thumb is to use cylindrical coordinates for everything other than spheres (this includes paraboloids, cylinders, and so on). If spheres are involved, then I use spherical coordinates.

Change of Variables

Suppose we want to integrate $f(x,y)$ over the region R, but R might be complicated in terms of x and y. If we can find a transformation $x = g(u,v), h(u,v)$ such that the region R becomes S, we can rewrite our integral as follows:

$$\iint_R f(x,y)dA = \iint_S f(g(u,v), h(u,v))|J|dudv$$

where J is the Jacobian ("fudge factor") given by

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

Surface Area

If the parametrization of a surface S is given by $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ where D is the region of u, v, then the surface area of S over D is

$$A = \iint_D ||\mathbf{r}_u \times \mathbf{r}_v|| dA.$$

Note that $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ is the normal to the surface S.

Surface Integrals

We can evaluate two types of surface integrals: surface integrals over scalar functions and surface integrals over vector fields.

If S is given by a parametrization $\mathbf{r}(u,v)$ and f is a scalar function defined on S, then

$$\iint_S f(x,y,z) dS = \iint_D f(\mathbf{r}(u,v)) ||\mathbf{r}_u \times \mathbf{r}_v|| dA$$

where D is the range of the parameters u, v.

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

If S is given by a vector function $\mathbf{r}(u,v)$, then the above integral can be written as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

where D is the parameter domain (the range of the parameters u, v).

If we are given S is the function $z = g(x,y)$, we can define the parametrization of S to be

$$\mathbf{r}(x,y) = (x, y, g(x,y)).$$

Then the surface integral of $\mathbf{F}$ over S is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) dA$$

where D is the projection of $z = g(x,y)$ onto the xy plane.

Stokes' Theorem

Theorem (Stokes' Theorem)**.** *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}.$$

In other words, the line integral around the boundary curve of a surface S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$.

Note: This looks very similar to Green's Theorem! In fact, Green's Theorem is a special case of Stokes' Theorem; it is when $\mathbf{F}$ is restricted to the xy plane. Stokes' Theorem can then be thought of as the higher-dimensional version of Green's Theorem.

Divergence Theorem

Theorem (Divergence Theorem)**.** *Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV.$$

In other words, the divergence theorem states that the flux of $\mathbf{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\mathbf{F}$ over E. This theorem allows us to change from a double integral to a triple integral.