Math 6A Practice Problems III

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Answers

1. -3
2. $171\sqrt{14}$
3. $\frac{\pi}{2}$
4. 0
5. (a) 80π
   (b) 80π
6. $\frac{9}{2}$
7. $\frac{32\pi}{3}$

Detailed Solutions

1. Use the transformation $x = 2u + v, y = u + 2v$ to evaluate the integral $\iint_R (x - 3y)dA$, where R is the triangular region with vertices $(0, 0), (2, 1), (1, 2)$.

   Solution. The equations of the lines (edges) of the triangle are $y = 2x, y = \frac{1}{2}x, y = 3 - x$. Plug the given transformations into these lines. When $y = 2x$:

   $$u + 2v = 2(2u + v) \quad \Rightarrow \quad u + 2v = 4u + 2v \quad \Rightarrow \quad u = 0.$$

   When $y = \frac{1}{2}x$:

   $$u + 2v = \frac{1}{2}(2u + v) \quad \Rightarrow \quad u + 2v = u + \frac{1}{2}v \quad \Rightarrow \quad v = 0.$$

   When $y = 3 - x$:

   $$u + 2v = 3 - (2u + v) \quad \Rightarrow \quad u + 2v = 3 - 2u - v \quad \Rightarrow \quad 3u + 3v = 3 \quad \Rightarrow \quad u + v = 1.$$

   The region that defines u, v can be described as $\{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1 - u\}$, or $\{(u, v) : 0 \leq v \leq 1, 0 \leq u \leq 1 - v\}$.

   We now calculate the Jacobian:

   $$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3.$$

   Therefore

   $$\begin{aligned} \iint_R (x - 3y)dA &= \int_0^1 \int_0^{1-u} [2u + v - 3(u + 2v)]|3|dvdu = 3 \int_0^1 \int_0^{1-u} (-u - 5v)dvdu \\ &= 3 \int_0^1 \left(-uv - \frac{5}{2}v^2\right) \Big|_{v=0}^{1-u} du = 3 \int_0^1 \left(-u(1 - u) - \frac{5}{2}(1 - u)^2\right) du \\ &= 3 \int_0^1 \left(-\frac{3}{2}u^2 + 4u - \frac{5}{2}\right) du = 3 \left(-\frac{1}{2}u^3 + 2u^2 - \frac{5}{2}u\right) \Big|_0^1 \\ &= -3 \end{aligned}$$

   □

2. Evaluate the surface integral $\iint_S x^2yzdS$ where S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$.

   Solution. $\iint_S x^2yzdS = \iint_D x^2yz||\mathbf{n}||dA$. We need to find a parametrization $\mathbf{r}$ to find the normal $\mathbf{n}$ and the region D. Start off with

   $$\mathbf{r} = \langle x, y, z \rangle.$$

   We need to narrow this down to two variables. Our surface is the plane $z = 1 + 2x + 3y$, plug this for z in our parametrization:

   $$\mathbf{r}(x, y) = \langle x, y, 1 + 2x + 3y \rangle.$$

   We are given that we are above the rectangle $[0, 3] \times [0, 2]$, which is $0 \leq x \leq 3, 0 \leq y \leq 2$. This is our region D.

   Now we calculate the normal vector $\mathbf{n} = \mathbf{r}_x \times \mathbf{r}_y$ and take the magnitude:

   $$\begin{aligned} \mathbf{r}_x &= \langle 1, 0, 2 \rangle \\ \mathbf{r}_y &= \langle 0, 1, 3 \rangle \\ \mathbf{n} = \mathbf{r}_x \times \mathbf{r}_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = \langle -2, -3, 1 \rangle \\ ||\mathbf{n}|| &= \sqrt{(-2)^2 + (-3)^1 + 1^2} = \sqrt{14}. \end{aligned}$$

Therefore

$$\begin{aligned}\iint_S x^2yz\,dS &= \int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{14}\,dydx = \sqrt{14}\int_0^3\int_0^2 (x^2y+2x^3y+3x^2y^2)dydx\\ &= \sqrt{14}\int_0^3\left(\frac{x^2y^2}{2}+x^3y^2+x^3y^3\right)\Big|_{y=0}^2 dx = \sqrt{14}\int_0^3(10x^2+4x^3)dx\\ &= \sqrt{14}\left(\frac{10}{3}x^3+x^4\right)\Big|_0^3 = 171\sqrt{14}.\end{aligned}$$

□

3. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = y\mathbf{i}+x\mathbf{j}+z\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1-x^2-y^2$ and the plane $z=0$.

Solution. You can also evaluate this surface integral using Divergence Theorem, but we will instead calculate the surface integral directly.

We have two surfaces: the paraboloid, call it S_1, and the plane $z=0$, call it S_2. We need to calculate the surface integral for each surface then add them together at the end:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}.$$

We will assume that S has positive (outward) orientation. This means that S_1 has upward orientation and S_2 has downward orientation.

Recall that $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot\mathbf{n}dA$ where $\mathbf{n}$ is the normal of our parametrization vector $\mathbf{r}$.

S_1: Start off with the parametrization

$$\mathbf{r} = \langle x,y,z\rangle$$

We need to narrow this down to two variables. Our surface is the paraboloid $z = 1-x^2-y^2$, plug this in for z in our parametrization:

$$\mathbf{r}(x,y) = \langle x,y,1-x^2-y^2\rangle.$$

The region D is the range of x,y, which we can find when the paraboloid intersects with the plane $z=0$. When $z=0$, we have the circle $x^2+y^2=1$, which can be described as $\{(r,\theta): 0\le r\le 1, 0\le\theta\le 2\pi\}$.

Now we calculate the normal vector $\mathbf{n} = \mathbf{r}_x\times\mathbf{r}_y$ and take the magnitude:

$$\begin{aligned}\mathbf{r}_x &= \langle 1,0,-2x\rangle\\ \mathbf{r}_y &= \langle 0,1,-2y\rangle\\ \mathbf{n} = \mathbf{r}_x\times\mathbf{r}_y &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -2x\\ 0 & 1 & -2y\end{vmatrix} = \langle 2x,2y,1\rangle\end{aligned}$$

Therefore

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \langle y,x,z\rangle\cdot\langle 2x,2y,1\rangle dA = \iint_D (4xy+z)dA = \iint_D(4xy+1-x^2-y^2)dA\\ &= \int_0^{2\pi}\int_0^1 (4r^2\cos\theta\sin\theta+1-r^2)r\,drd\theta = \int_0^{2\pi}\int_0^1(2r^3\sin 2\theta + r - r^3)drd\theta\\ &= \int_0^{2\pi}\left(\frac{r^4}{2}\sin 2\theta+\frac{r^2}{4}-\frac{r^4}{4}\right)\Big|_0^1 d\theta = \int_0^{2\pi}\left(\frac{\sin 2\theta}{2}+\frac{1}{4}\right)d\theta = -\frac{\cos 2\theta}{4}+\frac{\theta}{4}\Big|_0^{2\pi} = \frac{\pi}{2}.\end{aligned}$$

S_2: We can use the parametrization $\mathbf{r} = \langle x, y, 0 \rangle$ to find the normal vector, but we can also do so by observation. Since S_2 has downward orientation, the normal vector to $z = 0$ is $\mathbf{n} = -\mathbf{k} = \langle 0, 0, -1 \rangle$. Then

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle y, x, z \rangle \cdot \langle 0, 0, -1 \rangle dA = \iint_D 0 dA = 0.$$

Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}.$$

□

4. Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = x^2z^2\mathbf{i} + y^2z^2\mathbf{j} + xyz\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$.

Solution. We need to find a parametrization $\mathbf{r}(t)$ to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. The closed curve of intersection of the paraboloid and cylinder is the circle $x^2 + y^2 = 4$ at $z = 4$, which can be parametrized by

$$\mathbf{r}(t) = (2\cos t, 2\sin t, 4), \quad 0 \le t \le 2\pi.$$

Then we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} (64\cos^2 t, 64\sin^2 t, 16\cos t \sin t) \cdot (-2\sin t, 2\cos t, 0) dt \\ &= \int_0^{2\pi} (-128 \sin t \cos^2 t + 128 \sin^2 t \cos t) dt = 128 \left(\frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t \right) \Big|_0^{2\pi} = 0.\end{aligned}$$

□

5. Consider the vector field $\mathbf{F}(x, y, z) = yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}$, where C is circle $x^2 + y^2 = 16, z = 5$ oriented counterclockwise when viewed from above.

   (a) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by finding an appropriate parametrization vector $\mathbf{r}(t)$.

   Solution. The parametrization for the circle $x^2 + y^2 = 16, z = 5$ is given by

   $$\mathbf{r}(t) = (4\cos t, 4\sin t, 5), \quad 0 \le t \le 2\pi.$$

   Then

   $$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} (20\sin t, 40\cos t, e^{16\cos t \sin t}) \cdot (-4\sin t, 4\cos t, 0) dt \\ &= \int_0^{2\pi} (-80\sin^2 t + 160\cos^2 t) dt = \int_0^{2\pi} (-40(1 - \cos 2t) + 80(1 + \cos 2t)) dt \\ &= \int_0^{2\pi} (40 + 120\cos 2t) dt = (40t + 60\sin 2t)\Big|_0^{2\pi} = 80\pi.\end{aligned}$$

   □

   (b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem, and verify it is equal to your solution in part (a).

   Solution. We need to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$. The normal to the surface is the normal to $z = 5$, which is $\mathbf{n} = \mathbf{k}$. The curl of F is given by

   $$\begin{aligned}\text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & e^{xy} \end{vmatrix} = \mathbf{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & e^{xy} \end{vmatrix} - \mathbf{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ yz & e^{xy} \end{vmatrix} + \mathbf{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yz & 2xz \end{vmatrix} \\ &= \mathbf{i}(xe^{xy} - 2x) - \mathbf{j}(ye^{xy} - y) + \mathbf{k}(2z - z).\end{aligned}$$

Then curl $\mathbf{F} \cdot \mathbf{n} = z$, but on our surface $z = 5$, hence

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D z dA = \iint_D 5 dA = 5 \cdot \pi \cdot 4^2 = 80\pi.$$

□

6. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x, y, z) = 3x\mathbf{i} + xy\mathbf{j} + 2xz\mathbf{k}$ where E is the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
Note: to verify the theorem is true you need to show that $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV$; that is, you need to calculate both integrals and show they are equal.

Solution. We will evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ on each of the different faces of the cube then take their sum.
On $x = 0$, $\mathbf{n} = -\mathbf{i}$, $D = \{(y, z) : 0 \leq y \leq 1, 0 \leq z \leq 1\}$ and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D -3x dA = \iint_D 0 dA = 0.$$

On $x = 1$, $\mathbf{n} = \mathbf{i}$, $D = \{(y, z) : 0 \leq y \leq 1, 0 \leq z \leq 1\}$ and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D 3x dA = \iint_D 3 dA = 3.$$

On $y = 0$, $\mathbf{n} = -\mathbf{j}$, $D = \{(x, z) : 0 \leq x \leq 1, 0 \leq z \leq 1\}$ and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D -xy dA = \iint_D 0 dA = 0.$$

On $y = 0$, $\mathbf{n} = \mathbf{j}$, $D = \{(x, z) : 0 \leq x \leq 1, 0 \leq z \leq 1\}$ and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D xy dA = \iint_D x dA = \int_0^1 \int_0^1 x dx dz = \frac{1}{2}.$$

On $z = 0$, $\mathbf{n} = -\mathbf{k}$, $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq z \leq 1\}$ and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D -2xz dA = \iint_D 0 dA = 0.$$

On $z = 1$, $\mathbf{n} = \mathbf{k}$, $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq z \leq 1\}$ and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D 2xz dA = \iint_D 2x dA = \int_0^1 \int_0^1 2x dx dy = 1.$$

Therefore

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 0 + 3 + 0 + \frac{1}{2} + 0 + 1 = \frac{9}{2}.$$

Now we will evaluate $\iiint_E \text{div } \mathbf{F} dV$ where $E = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$. The divergence of F is given by

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(2xz) = 3 + x + 2x = 3x + 3.$$

Therefore

$$\iiint_E \text{div } \mathbf{F} dV = \iiint_E (3x + 3) dV = \int_0^1 \int_0^1 \int_0^1 (3x + 3) dx dy dz = \left(\frac{3}{2}x^2 + 3x\right)\Big|_0^1 = \frac{3}{2} + 3 = \frac{9}{2}.$$

□

7. Use the Divergence Theorem to calculate the surface integral $\iint \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S where $\mathbf{F}(x, y, z) = (\cos z + xy^2)\mathbf{i} + xe^{-z}\mathbf{j} + (\sin y + x^2 z)\mathbf{k}$, and S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Solution. We will evaluate $\iiint_E \text{div } \mathbf{F} dV$ where $E = \{(x, y, z) : x^2 + y^2 \leq z \leq 4\} = \{(r, \theta, z) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, r^2 \leq z \leq 4\}$. The divergence of F is given by

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(\cos z + xy^2) + \frac{\partial}{\partial y}(xe^{-z}) + \frac{\partial}{\partial z}(\sin y + x^2 z) = y^2 + x^2.$$

Then

$$\begin{aligned}\iiint_E \text{div } \mathbf{F} dV = \iiint_E (x^2 + y^2) dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \cdot r dz dr d\theta = 2\pi \int_0^2 \int_0^{2\pi} r^3 dz dr = 2\pi \int_0^2 r^3 z \Big|_{z=r^2}^{z=4} dr \\ = 2\pi \int_0^2 r^3(4 - r^2) dr = 2\pi \int_0^2 (4r^3 - r^5) dr = 2\pi \left(r^4 - \frac{1}{6} r^6 \right) \Big|_0^2 = 2\pi \left(16 - \frac{64}{6} \right) \\ = \frac{32\pi}{3}.\end{aligned}$$

□