Math 6A Practice Problems II

Written by Victoria Kala
vtkala@math.ucsb.edu
SH 6432u Office Hours: R 12:30 − 1:30pm
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Answers

This page contains answers only. Detailed solutions are on the following pages.

1. $\sqrt{5}\pi$
2. π
3. (a) Not conservative
   (b) Conservative
   $f = x^2y^3 + x\ln y + C$
4. (a) $\text{curl}\mathbf{F} = \mathbf{0}$
   (b) $f = xy^2\cos z + C$
   (c) 0
5. (a) 44
   (b) 88
   (c) $\dfrac{280}{3}$
6. (a) $\frac{21}{2}\ln 2$
   (b) $\frac{8}{15}(2\sqrt{2}-1)$
7. $\dfrac{3}{10}$
8. $\frac{1}{3}\left(2\sqrt{2}-1\right)$
9. $-\dfrac{14}{3}$
10. $\dfrac{1}{6}$
11. $\dfrac{14}{3}$
12. 144
13. $\dfrac{1}{8}+\dfrac{2}{21}-\dfrac{1}{14}-\dfrac{1}{24}$
14. $\dfrac{2}{35}$
15. $\pi(e^6 - e - 5)$
16. 0

Detailed Solutions

1. Evaluate the line integral $\int_C xyzds$ where C is the curve parametrized by $x = 2\sin t, y = t, z = -2\cos t$, $0 \le t \le \pi$.

   Solution. Since $\mathbf{r}(t) = \langle 2\sin t, t, -2\cos t\rangle$, $\mathbf{r}'(t) = \langle 2\cos t, 1, 2\sin t\rangle$ and

   $$||\mathbf{r}'(t)|| = \sqrt{(2\cos t)^2 + 1 + (2\sin t)^2} = \sqrt{5}.$$

   Then

   $$\int_C xyzds = \int_0^\pi (2\sin t)(t)(-2\cos t)||\mathbf{r}'(t)||dt = -4\sqrt{5}\int_0^\pi t\sin t\cos tdt = -2\sqrt{5}\int_0^\pi t\sin 2tdt$$
   $$= -2\sqrt{5}\left(-\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t\right)\Big|_0^\pi = \sqrt{5}\pi.$$

   We used integration by parts to evaluate the integral. □

2. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is given by the vector function $\mathbf{r}(t) = t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}$, $0 \le t \le \pi$ and $\mathbf{F} = z\mathbf{i} + y\mathbf{j} - x\mathbf{k}$.

   Solution. $\mathbf{F}(\mathbf{r}(t)) = \langle \cos t, \sin t, -t\rangle$ and $\mathbf{r}(t) = \langle 1, \cos t, -\sin t\rangle$. Therefore

   $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_0^\pi \langle \cos t, \sin t, -t\rangle\cdot\langle 1, \cos t, -\sin t\rangle dt$$
   $$= \int_0^\pi (\cos t + \sin t\cos t + t\sin t)\,dt = \int_0^\pi \left(\cos t + \frac{1}{2}\sin 2t + t\sin t\right)dt$$
   $$= \sin t - \frac{1}{4}\cos 2t - t\cos t + \sin t\Big|_0^\pi = \pi.$$

   We used integrating by parts to evaluate the integral $\int t\sin tdt$. □

3. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x,y) = e^x\cos y\mathbf{i} + e^x\sin y\mathbf{j}$

   Solution. $P = e^x\cos y$ and $Q = e^x\sin y$. Then

   $$\frac{\partial P}{\partial y} = -e^x\sin y \quad \text{and} \quad \frac{\partial Q}{\partial x}e^x\sin y.$$

   Since $\frac{\partial Q}{\partial x} \ne \frac{\partial P}{\partial y}$, $\mathbf{F}$ is not conservative. □

   (b) $\mathbf{F}(x,y) = (\ln y + 2xy^3)\mathbf{i} + \left(3x^2y^2 + \frac{x}{y}\right)\mathbf{j}$

Solution. $P = \ln y + 2xy^3$ and $Q = 3x^2y^2 + \frac{x}{y}$. Then

$$\frac{\partial P}{\partial y} = \frac{1}{y} + 6xy^2, \quad \text{and} \quad \frac{\partial Q}{\partial x} = 6xy^2 + \frac{1}{y}.$$

Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, $\mathbf{F}$ is conservative and there exists an f such that $\nabla f = \mathbf{F}$.
We wish to find a function f such that $\langle f_x, f_y \rangle = \langle \ln y + 2xy^3, 3x^2y^2 + \frac{x}{y} \rangle$:

$$\begin{aligned} f_x &= \ln y + 2xy^3 \quad &\Rightarrow \quad f &= x \ln y + x^2y^3 + g(y) \\ f_y &= 3x^2y^2 + \frac{x}{y} \quad &\Rightarrow \quad f &= x^2y^3 + x \ln y + h(x). \end{aligned}$$

Therefore $f = x^2y^3 + x \ln y + C$. □

4. Let $\mathbf{F}(x, y, z) = y^2 \cos z\mathbf{i} + 2xy \cos z\mathbf{j} - xy^2 \sin z\mathbf{k}$, C be the curve parametrized by $\mathbf{r}(t) = t^2\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq \pi$.

   (a) Show that $\mathbf{F}$ is conservative. (Hint: use curl)

   Solution. We need to show $\text{curl}(\mathbf{F}) = \mathbf{0}$:

$$\begin{aligned} \text{curl}(\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos z & 2xy \cos z & -xy^2 \sin z \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy \cos z & -xy^2 \sin z \end{vmatrix} - \mathbf{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y^2 \cos z & -xy^2 \sin z \end{vmatrix} + \mathbf{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^2 \cos z & 2xy \cos z \end{vmatrix} \\ &= \mathbf{i} \left(\frac{\partial}{\partial y}(-xy^2 \sin z) - \frac{\partial}{\partial z}(2xy \cos z) \right) - \mathbf{j} \left(\frac{\partial}{\partial x}(-xy^2 \sin z) - \frac{\partial}{\partial z}(y^2 \cos z) \right) \\ &\quad + \mathbf{k} \left(\frac{\partial}{\partial x}(2xy \cos z) - \frac{\partial}{\partial y}(y^2 \cos z) \right) \\ &= (-2xy \sin z + 2xy \sin z)\mathbf{i} - (-y^2 \sin z + y^2 \sin z)\mathbf{j} + (2y \cos z - 2y \cos z)\mathbf{k} \\ &= \mathbf{0}. \end{aligned}$$

   Therefore $\mathbf{F}$ is conservative. □

   (b) Find a function f such that $\mathbf{F} = \nabla f$.

   Solution. We need to find an f such that $\langle f_x, f_y, f_z \rangle = \langle y^2 \cos z, 2xy \cos z, -xy^2 \sin z \rangle$:

$$\begin{aligned} f_x &= y^2 \cos z \quad &\Rightarrow \quad f &= xy^2 \cos z + g(y, z) \\ f_y &= 2xy \cos z \quad &\Rightarrow \quad f &= xy^2 \cos z + h(x, z) \\ f_z &= -xy^2 \sin z \quad &\Rightarrow \quad f &= xy^2 \cos z + \ell(x, y) \end{aligned}$$

   Therefore $f = xy^2 \cos z + C$. □

   (c) Use (b) to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

Solution.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0) = f(\pi^2, 1, \pi) - f(0,0,0) = 0.$$

□

5. (a) Estimate the volume of the solid that lies below the surface $z = x + 2y^2$ and above the rectangle $R = [0,2]\times[0,4]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be the lower right corners.

Solution. $\Delta x = \frac{2-0}{2} = 1$ and $\Delta y = \frac{4-0}{2}$, therefore $\Delta A = \Delta x \Delta y = 2$. The lower righthand corners of each rectangle are $(1,0),(1,2),(2,0),(2,2)$. Let $z = f(x,y)$. Then

$$\iint f(x,y)dA \approx (f(1,0) + f(1,2) + f(2,0) + f(2,2))\,\Delta A = (1+9+2+10)\cdot 2 = 44.$$

□

(b) Use the midpoint rule to estimate the volume in (a).

Solution. We did not change the division of our rectangles, so $\Delta A = 2$. The midpoints of each rectangle are $(\frac{1}{2},1),(\frac{1}{2},3),(\frac{3}{2},1),(\frac{3}{2},3)$. Then

$$\begin{aligned}\iint f(x,y)dA &\approx \left(f\left(\frac{1}{2},1\right) + f\left(\frac{1}{2},3\right) + f\left(\frac{3}{2},1\right) + f\left(\frac{3}{2},3\right)\right)\Delta A \\ &= \left(\frac{1}{2} + 2 + \frac{1}{2} + 18 + \frac{3}{2} + 2 + \frac{3}{2} + 18\right)\cdot 2 = 88.\end{aligned}$$

□

(c) Calculate the exact volumes of the solid.

Solution.

$$\int_0^2\int_0^4 (x+2y^2)dydx = \int_0^2 \left(xy + \frac{2}{3}y^3\right)\Big|_0^4 dx = \int_0^2 \left(4x + \frac{128}{3}\right)dx = 2x^2 + \frac{128}{3}x\Big|_0^2 = \frac{280}{3}$$

□

6. Evaluate the following integrals:

(a) $\displaystyle\int_1^4\int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right)dydx$

Solution.

$$\begin{aligned}\int_1^4\int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right)dydx &= \int_1^4 \left(x\ln y + \frac{y^2}{2x}\right)\Big|_{y=1}^2 dx = \int_1^4 \left(x\ln 2 + \frac{2}{x} - \frac{1}{2x}\right)dx \\ &= \frac{1}{2}x^2\ln 2 + 2\ln x - \frac{1}{2}\ln x\Big|_1^4 = \frac{11}{2}\ln 2 - \frac{3}{2}\ln 4 = \frac{21}{2}\ln 2\end{aligned}$$

□

(b) $\int_0^1 \int_0^1 \sqrt{s+t}\ dsdt$

Solution.

$$\begin{aligned}\int_0^1 \int_0^1 (s+t)^{1/2}\ dsdt &= \int_0^1 \frac{2}{3}(s+t)^{3/2}\Big|_0^1 dt = \frac{2}{3}\int_0^1 \left((1+t)^{3/2} - t^{3/2}\right) dt \\ &= \frac{2}{3}\left(\frac{2}{5}(1+t)^{5/2} - \frac{2}{5}t^{5/2}\right)\Big|_0^1 = \frac{8}{15}(2\sqrt{2}-1)\end{aligned}$$

□

7. Evaluate $\iint_D (x+y)\ dA$ where D is bounded by $y=\sqrt{x}, y=x^2$.

Solution. The region described is $D=\{(x,y) : 0\le x\le 1, x^2 \le y \le \sqrt{x}\}$.

$$\begin{aligned}\iint_D (x+y)dA &= \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)dydx = \int_0^1 \left(xy+\frac{y^2}{2}\right)\Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 \left(x^{3/2}+\frac{x}{2}-x^3-\frac{x^4}{2}\right) dx \\ &= \frac{2}{5}x^{5/2}+\frac{x^2}{4}-\frac{x^4}{4}-\frac{x^5}{5}\Big|_0^1 = \frac{3}{10}\end{aligned}$$

□

8. Evaluate the integral by reversing the order of integration:

$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}\ dxdy.$$

Solution. The region described in the integral above is equivalent to $0\le x\le \frac{\pi}{2}, 0\le y\le \sin x$:

$$\begin{aligned}\int_0^{\pi/2} \int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\ dydx &= \int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\cdot y\Big|_0^{\sin x} dx \\ &= \int_0^{\pi/2} \sin x\cos x\sqrt{1+\cos^2 x}dx, \qquad \text{let } u = 1+\cos^2 x \\ &= -\frac{1}{2}\int_2^1 \sqrt{u}du = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_2^1 = \frac{1}{3}\left(2\sqrt{2}-1\right)\end{aligned}$$

□

9. Evaluate $\iint_R (x+y)\ dA$ where R is the region that lies to the left of the y-axis between the circles $x^2+y^2=1, x^2+y^2=4$.

Solution. The region described in polar coordinates is $R = \{(r,\theta) : 1 \le r \le 2, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$.

$$\begin{aligned}\iint_R (x+y)dA &= \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta) r dr d\theta = \int_{\pi/2}^{3\pi/2}\int_1^2 (\cos\theta+\sin\theta) r^2 dr d\theta \\ &= \int_{\pi/2}^{3\pi/2} (\cos\theta+\sin\theta)\cdot \frac{r^3}{3}\Big|_1^2 d\theta = \frac{7}{3}\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)d\theta \\ &= \frac{7}{3}(\sin\theta-\cos\theta)\Big|_{\pi/2}^{3\pi/2} = -\frac{14}{3}\end{aligned}$$

□

10. Verify Green's Theorem for $\int_C x^4 dx + xy dy$ where C is the triangular curve consisting of line segments from $(0,0)$ to $(1,0)$, $(1,0)$, to $(0,1)$, and $(0,1)$ to $(0,0)$ traversed in that order.

Solution. We need to verify $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$. We need to calculate both the left integral and the right integral and show they are equal.

For the left integral: We need to break up our triangle into three different sections. Let C_1 be the side of the triangle from $(0,0)$ to $(1,0)$, C_2 be the side of the triangle from $(1,0)$ to $(0,1)$, and C_3 be the side of the triangle from $(0,1)$ to $(0,0)$. Then $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$.

C_1 is parametrized by $\mathbf{r}(t) = \langle t, 0\rangle$, $0 \le t \le 1$. Then

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 F(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt = \int_0^1 t^4 dt + t\cdot 0\cdot 0dt = \int_0^1 t^4 dt = \frac{1}{5}t^5\Big|_0^1 = \frac{1}{5}.$$

C_2 is parametrized by $\mathbf{r}(t) = \langle 1-t, t\rangle, 0 \le t \le 1$. Then

$$\begin{aligned}\int_{C_2} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 F(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt = \int_0^1 (1-t)^4(-dt) + (1-t)t dt = \int_0^1 \left(-(1-t)^4 + t - t^2\right) dt \\ &= \frac{1}{5}(1-t)^5 + \frac{1}{2}t^2 - \frac{1}{3}t^3\Big|_0^1 = \frac{1}{2} - \frac{1}{3} - \frac{1}{5}\end{aligned}$$

C_3 is parametrized by $\mathbf{r}(t) = \langle 0, 1-t\rangle, 0 \le t \le 1$. Then

$$\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 F(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt = \int_0^1 0(-dt) + 0\cdot(1-t)(-dt) = 0.$$

Therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{5} + \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + 0 = \frac{1}{6}.$$

Clearly this method is tedious. Now we will apply Green's Theorem. The triangle can be described as $D = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1-x\}$. Then

$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \int_0^1\int_0^{1-x} \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^4)\right) dy dx = \int_0^1\int_0^{1-x} y dy dx \\ &= \int_0^1 \frac{y^2}{2}\Big|_0^{1-x} dx = \int_0^1 \frac{(1-x)^2}{2} dx = -\frac{(1-x)^3}{6}\Big|_0^1 = \frac{1}{6}.\end{aligned}$$

Our two calculations are equal to each other, thus the theorem is verified. □

11. Evaluate $\int_C y^2 dx + 3xydy$, where C is the boundary of the semiannular region D in the upper half plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$. (You may assume that C is positively oriented.)

Solution. D is the region $\{(x,y) : 1 \le x^2+y^2 \le 4\} = \{(r,\theta) : 1 \le r \le 2, 0 \le \theta \le \pi\}$. Using Green's Theorem, we have

$$\begin{aligned}\int_C y^2 dx + 3xydy &= -\iint_D \left(\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right) dA = \iint_D (3y-2y))\, dA = \iint_D ydA \\ &= \int_0^\pi \int_1^2 r\sin\theta \cdot rdrd\theta = \int_0^\pi \sin\theta d\theta \int_1^2 r^2 dr = (-\cos\theta)\Big|_0^\pi \cdot \frac{1}{3}r^3\Big|_1^2 \\ &= \frac{14}{3}\end{aligned}$$

□

12. Use a triple integral to find the volume of the solid bounded by the cylinder $y = x^2$ and the planes $z=0, z=4$, and $y=9$.

Solution. The solid can be described as $E = \{(x,y,z) : -3 \le x \le 3, x^2 \le x \le 9, 0 \le z \le 4\}$. Therefore

$$\begin{aligned}V &= \iiint_E dV = \int_{-3}^3 \int_{x^2}^9 \int_0^4 dzdydx = \int_{-3}^3 \int_{x^2}^9 z\Big|_0^4 dydx = 4\int_{-3}^3 \int_{x^2}^9 dydx = 4\int_{-3}^3 y\Big|_{x^2}^9 dx \\ &= 4\int_{-3}^3 (9-x^2)dx = 4\left(9x - \frac{x^3}{3}\right)\Big|_{-3}^3 = 144\end{aligned}$$

□

13. Evaluate $\iiint_E xydV$ where E is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$, and the planes $z = 0$ and $z = x + y$.

Solution. The solid can be described as $E = \{(x,y,z) : 0 \le x \le 1, x^2 \le y \le \sqrt{x}, 0 \le z \le x+y\}$. Then

$$\begin{aligned}\iiint_E xydV &= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xydzdydx = \int_0^1 \int_{x^2}^{\sqrt{x}} xyz\Big|_{z=0}^{x+y} dydx = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2y + xy^2)dydx \\ &= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)\Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 \left(\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right) dx \\ &= \frac{x^4}{8} + \frac{2x^{7/2}}{21} - \frac{x^7}{14} - \frac{x^8}{24}\Big|_0^1 = \frac{1}{8} + \frac{2}{21} - \frac{1}{14} - \frac{1}{24}\end{aligned}$$

□

14. Evaluate $\iiint_E (x^3 + xy^2)dV$ where E is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

Solution. The solid can be described in cylindrical coordinates as $E = \{(r, \theta, z) : 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq z \leq 1 - r^2\}$. Since $x^3 + xy^2 = x(x^2 + y^2) = r\cos\theta \cdot r^2$, then

$$\begin{aligned}\iiint_E (x^3 + xy^2)dV &= \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r^3\cos\theta \cdot r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^4\cos\theta \cdot z\Big|_0^{1-r^2} dr\,d\theta \\ &= \int_0^{\pi/2}\int_0^1 r^4(1-r^2)\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 (r^4 - r^6)\cos\theta\,dr\,d\theta \\ &= \int_0^{\pi/2}\left(\frac{r^5}{5} - \frac{r^7}{7}\right)\Big|_0^1 \cos\theta\,d\theta = \frac{2}{35}\sin\theta\Big|_0^{\pi/2} = \frac{2}{35}\end{aligned}$$

□

15. Evaluate $\iiint_E e^z dV$ where E is enclosed by the paraboloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$, and the xy-plane.

Solution. The solid can be described in cylindrical coordinates as $E = \{(r, \theta, z) : 0 \leq r \leq \sqrt{5}, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1 + r^2\}$.

$$\begin{aligned}\iiint_E e^z dV &= \int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{1+r^2} e^z r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}} re^z\Big|_{z=0}^{1+r^2} dr\,d\theta \\ &= \int_0^{2\pi}\int_0^{\sqrt{5}} r(e^{1+r^2} - 1)\,dr\,d\theta = 2\pi\left(\int_0^{2\pi} re^{1+r^2}dr - \int_0^{2\pi} r\,dr\right), \qquad \text{let } u = 1 + r^2 \\ &= 2\pi\left(\frac{1}{2}\int_1^6 e^u du - \frac{r^2}{2}\Big|_0^{\sqrt{5}}\right) = \pi\left(e^u\Big|_1^6 - 5\right) = \pi(e^6 - e - 5)\end{aligned}$$

□

16. Evaluate $\iiint_E xyz\,dV$ where E lies between the spheres $\rho = 2$ and $\rho = 4$ and the cone $\phi = \frac{\pi}{3}$.

Solution. The solid can be described in polar coordinates as $E = \{(\rho, \theta, \phi) : 2 \leq \rho \leq 4, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{3}\}$. Then

$$\begin{aligned}\iiint_E xyz\,dV &= \int_2^4\int_0^{2\pi}\int_0^{\pi/3} (\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)(\rho\cos\phi)\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \int_2^4 \rho^5 d\rho \int_0^{2\pi}\sin\theta\cos\theta\,d\theta\int_0^{\pi/3}\sin^3\phi\cos\phi\,d\phi \\ &= \frac{\rho^6}{6}\Big|_2^4 \cdot \frac{1}{2}\sin^2\theta\Big|_0^{2\pi} \cdot \frac{1}{4}\sin^4\phi\Big|_0^{\pi/3} \\ &= 0\end{aligned}$$

□