Math 6B: PDE "Quiz" Solution

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1. Vibration of a stretched string with fixed ends: The ends of a stretched string of length $L = 1$ are fixed at $x = 0$ and $x = 1$. The string is set to vibrate from rest by releasing it from an initial triangular shape model by the function

$$f(x) = \begin{cases} \frac{3}{10}x, & 0 \le x \le \frac{1}{3} \\ \frac{3(1-x)}{20}, & \frac{1}{3} \le x \le 1. \end{cases}$$

Determine the subsequent motion of the string (find $u(x,t)$) given that $c = 1/\pi$.

Hint: You need to solve the system $\dfrac{\partial^2 u}{\partial^2 t} = \dfrac{1}{\pi^2}\dfrac{\partial^2 u}{\partial x^2}, u(0,t) = u(1,t) = 0, u(x,0) = f(x), u_t(x,0) = 0, 0 < x < 1, t > 0.$

Solution. Given $L = 1, c = \frac{1}{\pi}$, the general solution to the equation is

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(b_n \cos\lambda_n t + b_n^* \sin\lambda_n t\right)$$

where

$$\begin{aligned} b_n &= 2\int_0^1 f(x)\sin n\pi x dx \\ b_n^* &= \frac{2}{n}\int_0^1 g(x)\sin n\pi x dx \\ \lambda_n &= n, \quad n = 1, 2, ... \end{aligned}$$

Find b_n (use integration by parts):

$$\begin{aligned} b_n &= 2\left[\int_0^{1/3} \frac{3}{10}x\sin(n\pi x) + \int_{1/3}^1 \frac{3(1-x)}{20}\sin(n\pi x)dx\right] \\ &= 2\left[\left(-\frac{1}{n\pi}\frac{3}{10}x\cos(n\pi x) + \frac{1}{(n\pi)^2}\frac{3}{10}\sin(n\pi x)dx\right)\Big|_0^{1/3} + \left(-\frac{1}{n\pi}\frac{3(1-x)}{20}\cos(n\pi x) - \frac{1}{(n\pi)^2}\frac{3}{20}\sin(n\pi x)\right)\Big|_{1/3}^1\right] \\ &= \frac{9}{10(n\pi)^2}\sin\left(\frac{n\pi}{3}\right) \end{aligned}$$

Find b_n^*:

$$b_n^* = \frac{2}{n}\int_0^1 0 \cdot \sin(n\pi x)dx = 0.$$

Substitute into the general solution:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{9}{10(n\pi)^2}\sin\left(\frac{n\pi}{3}\right)\sin(n\pi x)\cos nt.$$

□

2. A thin bar of length π units is placed in boiling water (temperature 100°C). After reaching 100°C throughout, the bar is removed from the boiling water. With the lateral sides kept insulated, suddenly, at time $t = 0$, the ends are immersed in a medium with constant freezing temperature 0°C. Taking $c = 1$, find the temperature $u(x, t)$ for $t > 0$.

   Hint: You need to solve the system $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, u(0, t) = u(\pi, t) = 0, u(x, 0) = 100, 0 < x < \pi, t > 0.$

   Solution. Given $L = \pi, c = 1$, the general solution to the equation is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin(nx)$$

   where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$
$$\lambda_n = n.$$

   Find b_n:

$$b_n = \frac{2}{\pi} \int_0^{\pi} 100 \sin(nx) dx = \frac{2}{\pi} \cdot 100 \cdot \frac{(-\cos(nx))}{n}\Big|_0^{\pi} = \frac{200}{\pi} \cdot \frac{1 - \cos(n\pi)}{n}$$

   When n is odd, $\cos(n\pi) = -1$ and $b_n = \dfrac{400}{n\pi}$. When n is even, $\cos(n\pi) = 1$ and $b_n = 0$. Substitute into the general solution:

$$u(x,t) = \sum_{n \text{ odd}} \frac{400}{n\pi} e^{-n^2 t} \sin(nx).$$

   We can do better. Since n is odd, $n = 2k + 1$ for $k = 0, 1, 2, ...$ Substitute this into our solution:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{400}{(2k+1)\pi} e^{-(2k+1)^2 t} \sin((2k+1)x).$$

□