Math 6B Practice Problems II

Written by Victoria Kala
vtkala@math.ucsb.edu
SH 6432u Office Hours: R 12:30 − 1:30pm
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Answers

This page contains answers only. Detailed solutions are on the following pages.

1. (a) $R = \infty, I = (-\infty, \infty)$
   (b) $R = 4,\ I = (-4, 4]$
   (c) $R = 0, I = \{\frac{1}{2}\}$

2. (a) $f(x) = -\sum_{n=0}^{\infty} x^{7n}$
   (b) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{2^{n+1}}$

3. $-\sum_{n=0}^{\infty} \frac{1}{3^{n+1}} (x+3)^n$

4. (a) $\sum_{n=0}^{\infty} \frac{1}{(2n+1)n!} x^{2n+1} + C$
   (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(6n+2)(2n)!} + C$

5. $\frac{1}{3}$

6. (a) e^{x^4}
   (b) $-1 + e^3$

7. (a) $f(x) = \frac{3}{2} - \sin x + 3\cos 2x - \frac{1}{2}\cos 6x$
   (b) $f(x) = \frac{p^2}{3} + \sum_{n=0}^{\infty} \frac{4p^2}{(n\pi)^2} (-1)^n \cos\left(\frac{n\pi}{p} x\right)$
   (c) $f(x) = \sum_{n=1}^{\infty} \frac{16}{\pi} (-1)^{n+1} \frac{n}{(2n+1)^2(2n-1)^2} \sin(2nx)$

8. Cosine series: $\frac{1}{2} + \sum_{k=0}^{\infty} \frac{-4}{(2k+1)^2\pi^2} \cos(2k+1)x$

   Sine series: $\sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi x)$

9. $u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \sin(n\pi x)\cos(4n\pi t)$

10. $u(x,t) = \sum_{n=1}^{\infty} -\frac{2n\pi}{(n\pi)^2 + 1} \left(\frac{(-1)^n}{e} + 1\right) e^{-(n\pi)^2 t} \sin(n\pi x)\,.$

Detailed Solutions

1. Find the radius and interval convergence of the series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

   Solution. This is a power series about $a = 0$. Using the Ratio Test with $a_n = \frac{x^n}{n!}$, we have

   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 < 1$$

   Since our limit is always less than 1, the radius of convergence is $R = \infty$ and the interval of convergence is $I = (-\infty, \infty)$. □

   (b) $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{4^n \ln n}$

   Solution. This is a power series about $a = 0$. Using the Ratio test with $a_n = (-1)^n \frac{x^n}{4^n \ln n}$, we have

   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{4^{n+1}\ln(n+1)}\cdot\frac{4^n\ln n}{(-1)^n x^n}\right| = \lim_{n\to\infty}\frac{|x|}{4}\frac{\ln n}{\ln(n+1)} \overset{L'H}{=} \lim_{n\to\infty}\frac{|x|}{4}\frac{n+1}{n} = \frac{|x|}{4} < 1$$

   "L'H" denotes where we used L'Hôpital's Rule to evaluate the limit. We have $|x| < 4$, hence the radius of convergence is $R = 4$. We need to test the endpoints at $x = -4, x = 4$.
   When $x = -4$:

   $$\sum_{n=2}^{\infty}(-1)^n\frac{(-4)^n}{4^n\ln n} = \sum_{n=2}^{\infty}\frac{1}{\ln n}.$$

   Since $n > \ln n$ (graph it), then $\frac{1}{\ln n} > \frac{1}{n}$. The sum $\sum\frac{1}{n}$ diverges, therefore by the Comparison Test, $\sum\frac{1}{\ln n}$ must also diverge. We do not include $x = -4$ in the interval of convergence.
   When $x = 4$:

   $$\sum_{n=2}^{\infty}(-1)^n\frac{(-4)^n}{4^n\ln n} \quad \sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}.$$

   The sequence $b_n = \frac{1}{\ln n}$ is decreasing and $\lim_{n\to\infty} b_n = 0$. Thus by the Alternating Series test, the series $\sum\frac{(-1)^n}{\ln n}$ converges. We include $x = 4$ in the interval of convergence.
   Therefore the interval of convergence is $(-4, 4]$. □

   (c) $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

   Solution. This is a power series about $a = \frac{1}{2}$ since $(2x-1)^n = [2(x-\frac{1}{2})]^n$. Using the ratio test with $a_n = n!(2x-1)^n$, we have

   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim_{n\to\infty}(n+1)|2x-1| = \infty.$$

   The limit is never less than 1, so the radius of convergence is $R = 0$. Therefore the interval of convergence is just the point $\{\frac{1}{2}\}$. □

2. Use the power series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to find the power series of the following functions:

(a) $f(x) = \frac{1}{x^7 - 1}$

Solution.

$$f(x) = -\frac{1}{1-x^7} = -\sum_{n=0}^{\infty}(x^7)^n = -\sum_{n=0}^{\infty} x^{7n}.$$

□

(b) $f(x) = \frac{x^3}{x+2}$

Solution.

$$f(x) = \frac{x^3}{2} \cdot \frac{1}{1+\frac{x}{2}} = \frac{x^3}{2} \cdot \frac{1}{1-(-\frac{x}{2})} = \frac{x^3}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+3}}{2^{n+1}}.$$

□

3. Find the Taylor series for $f(x) = \frac{1}{x}$ centered at $a = -3$.

Solution. The Taylor series of $f(x)$ about $a = -3$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Find a pattern for the nth derivative:

$$\begin{aligned}
f(x) &= x^{-1} \\
f'(x) &= (-1)x^{-2} \\
f''(x) &= (-1)(-2)x^{-3} \\
f'''(x) &= (-1)(-2)(-3)x^{-4} \\
&\vdots \\
f^{(n)}(x) &= (-1)(-2)\cdots(-n)x^{-(n+1)} = (-1)^n n! x^{-(n+1)}
\end{aligned}$$

Evaluate the nth derivative at $a = -3$:

$$f^{(n)}(-3) = (-1)^n n!(-3)^{-(n+1)} = \frac{(-1)^n n!}{(-1)^{n+1}3^{n+1}} = \frac{(-1)n!}{3^{n+1}}.$$

Therefore the Taylor series about $a = -3$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{(-1)n!}{3^{n+1}n!}(x+3)^n = -\sum_{n=0}^{\infty} \frac{1}{3^{n+1}}(x+3)^n.$$

□

4. Evaluate the indefinite integral as an infinite series:

(a) $\int e^{x^2}\,dx$

Solution. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$,

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Then

$$\int e^{x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\,dx = \int \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + ...\right) dx = x + \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + ... + C$$
$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)n!} x^{2n+1} + C$$

□

(b) $\int x\cos(x^3)dx$

Solution. Since $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$,

$$\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}.$$

Then

$$\int x\cos(x^3)dx = \int x \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} dx = \int x \left(1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + ...\right) dx$$
$$= \int \left(x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + ...\right) dx = \frac{x^2}{2} - \frac{x^8}{8 \cdot 2!} + \frac{x^{14}}{14 \cdot 4!} - \frac{x^{20}}{20 \cdot 6!} + ... + C$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(6n+2)(2n)!} + C$$

□

5. Use series to evaluate the limit $\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3}$.

Solution. Since $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, then

$$\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x \to 0} \frac{x - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}{x^3} = \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{x^3}{3} - \frac{x^5}{5!} + \frac{x^7}{7!} + ...}{x^3} = \lim_{x \to 0} \frac{1}{3} - \frac{x^2}{5!} + \frac{x^4}{7!} + ... = \frac{1}{3}.$$

□

6. Find the sum of the series:

(a) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{n!}$

Solution. Rewrite the series as

$$\sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}.$$

□

(b) $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + ...$

Solution. We can write the sum as the series

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} = -\frac{3^0}{0!} + \frac{3^0}{0!} + \sum_{n=1}^{\infty} \frac{3^n}{n!} = -1 + \sum_{n=0}^{\infty} \frac{3^n}{n!} = -1 + e^3.$$

□

7. Find the Fourier series of the function:

(a) $f(x) = 1 - \sin x + 3\cos 2x + \sin^2(3x)$, $-\pi \leq x \leq \pi$

Solution. A Fourier series is a sum of cosine and sine terms. We already have cosine and sine terms here, we just need to rewrite $\sin^2(3x)$ and we are done:

$$\begin{aligned} f(x) &= 1 - \sin x + 3\cos 2x + \sin^2(3x) = 1 - \sin x + 3\cos 2x + \frac{1}{2}(1 - \cos 6x) \\ &= \frac{3}{2} - \sin x + 3\cos 2x - \frac{1}{2}\cos 6x \end{aligned}$$

□

(b) $f(x) = x^2$, $-p \leq x \leq p$

Solution. f is even so it only has cosine terms in its Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right).$$

Since we are on the interval $-p \leq x \leq p$, $L = p$. Find a_0 and a_n (use integration by parts for a_n):

$$\begin{aligned} a_0 &= \frac{1}{p}\int_0^p f(x)dx = \frac{1}{p}\int_0^p x^2 dx = \frac{x^3}{3p}\Big|_0^p = \frac{p^2}{3} \\ a_n &= \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)dx = \frac{2}{p}\int_0^p x^2\cos\left(\frac{n\pi}{p}x\right)dx \\ &= \frac{2}{p}\left[x^2\frac{p}{n\pi}\sin\left(\frac{n\pi}{p}x\right) + 2x\left(\frac{p}{n\pi}\right)^2\cos\left(\frac{n\pi}{p}x\right) - 2\left(\frac{p}{n\pi}\right)^3\sin\left(\frac{n\pi}{p}x\right)\right]\Big|_0^p \\ &= \frac{4p^2}{(n\pi)^2}\cos n\pi = \frac{4p^2}{(n\pi)^2}(-1)^n. \end{aligned}$$

Thus

$$f(x) = \frac{p^2}{3} + \sum_{n=0}^{\infty} \frac{4p^2}{(n\pi)^2}(-1)^n\cos\left(\frac{n\pi}{p}x\right).$$

□

(c) $f(x) = x\cos x$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$
Hint: This is a tough one. Use the fact that the function is odd. Also use the formula $\cos(\alpha)\sin(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$.

Solution. f is odd, so it only has sine terms in its Fourier series:

$$f(x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{L}x\right).$$

Since we are on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $L = \frac{\pi}{2}$. Find b_n:

$$b_n = \frac{2}{\frac{\pi}{2}} \int_0^{\pi/2} f(x)\sin\left(\frac{n\pi}{\frac{\pi}{2}}x\right)dx = \frac{4}{\pi}\int_0^{\pi/2} x\cos x \sin(2nx)dx$$

Use the given formula on $\cos x \sin(2nx)$, then evaluate using integration by parts:

$$\begin{aligned}
b_n &= \frac{4}{\pi}\int_0^{\pi/2} x\cos x\sin(2nx)dx = \frac{4}{\pi}\int_0^{\pi/2} x\cdot\frac{1}{2}\left(\sin(2n+1)x + \sin(2n-1)x\right)dx \\
&= \frac{2}{\pi}\left[x\left(-\frac{\cos(2n+1)x}{2n+1} - \frac{\cos(2n-1)x}{2n-1}\right) + \frac{\sin(2n+1)x}{(2n+1)^2} + \frac{\sin(2n-1)x}{(2n-1)^2}\right]\Bigg|_0^{\pi/2} \\
&= \frac{2}{\pi}\left[\frac{\sin(2n+1)\frac{\pi}{2}}{(2n+1)^2} + \frac{\sin(2n-1)\frac{\pi}{2}}{(2n-1)^2}\right] = \frac{2}{\pi}\left[\frac{(-1)^{n+1}}{(2n+1)^2} + \frac{(-1)^n}{(2n-1)^2}\right] \\
&= \frac{2}{\pi}(-1)^n\left(\frac{-1}{(2n+1)^2} + \frac{1}{(2n-1)^2}\right) = \frac{2}{\pi}(-1)^n\left(\frac{(2n+1)^2 - (2n-1)^2}{(2n+1)^2(2n-1)^2}\right) \\
&= \frac{16}{\pi}(-1)^{n+1}\frac{n}{(2n+1)^2(2n-1)^2}.
\end{aligned}$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \frac{16}{\pi}(-1)^{n+1}\frac{n}{(2n+1)^2(2n-1)^2}\sin(2nx).$$

□

8. Find the cosine and sine series of the function $f(x) = x$, $0 < x < 1$.

Solution. We are only given half of our periodic function.

For the cosine series, we need to find the even expansion of $f(x) = x$ which is $f(x) = |x|$ for $-1 < x < 1$. In this case, $L = 1$. The cosine series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi x)$$

where

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx = \int_0^1 xdx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}. \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos(n\pi x)dx = 2\int_0^1 x\cos(n\pi x)dx = 2\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]\Bigg|_0^1 \\
&= \frac{2}{(n\pi)^2}(\cos(n\pi) - 1).
\end{aligned}$$

When n is even, $\cos(n\pi) = 1$ which implies that $a_n = 0$. When n is odd, $\cos(n\pi) = -1$, which implies that

$$a_n = \frac{-4}{(n\pi)^2}.$$

Thus the cosine series is

$$f(x) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{-4}{(n\pi)^2}\cos(n\pi x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{-4}{(2k+1)^2\pi^2}\cos(2k+1)x.$$

For the sine series, we need to find the odd expansion of $f(x)$ which is $f(x) = x$ for $-1 < x < 1$. In this case, $L = 1$. The sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin(n\pi x) = 2\int_0^1 x\sin(n\pi x)dx = 2\left[\frac{-x}{n\pi}\cos(n\pi x) + \frac{1}{(n\pi)^2}\sin(n\pi x)\right]\Big|_0^1 \\ &= -\frac{2}{n\pi}\cos(n\pi) = -\frac{2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}. \end{aligned}$$

Thus the sine series is

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x).$$

□

9. Solve the system $u_{tt} = 16u_{xx}, 0 \le x \le 1, t \ge 0; u(0,t) = u(1,t) = 0; u(x,0) = f(x), u_t(x,0) = 0$ where

$$f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \le 1. \end{cases}$$

Solution. This is a wave equation system. We are given $c = 4$, $L = 1$, $g(x) = 0$, and $f(x)$ as described above. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(b_n \cos\lambda_n t + b_n^* \sin\lambda_n t\right)$$

where

$$\begin{aligned} b_n &= 2\int_0^1 f(x)\sin(n\pi x)dx \\ b_n^* &= \frac{2}{4n\pi}\int_0^1 g(x)\sin(n\pi x)dx \\ \lambda_n &= 4n\pi. \end{aligned}$$

$b_n^* = 0$ since $g(x) = 0$. Find b_n (use integration by parts):

$$\begin{aligned} b_n &= 2\left[\int_0^{1/2} 2x\sin(n\pi x)dx + \int_{1/2}^1 2(1-x)\sin(n\pi x)dx\right] \\ &= 2\left[\left(-\frac{2x}{n\pi}\cos(n\pi x) + \frac{2}{(n\pi)^2}\sin(n\pi x)\right)\Big|_0^{1/2} + \left(-\frac{2(1-x)}{n\pi}\cos(n\pi x) - \frac{2}{(n\pi)^2}\sin(n\pi x)\right)\Big|_{1/2}^1\right] \\ &= \frac{8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\sin(n\pi x)\cos(4n\pi t).$$

You could clean this up more if you want to. When n is even $\sin\left(\frac{n\pi}{2}\right) = 0$, so only the odd terms contribute. When $n = 4k+1$, $\sin\left(\frac{n\pi}{2}\right) = 1$. When $n = 4k+3$, $\sin\left(\frac{n\pi}{2}\right) = -1$. □

10. Solve the system $u_t = u_{xx}, 0 \le x \le 1, t \ge 0; u(0,t) = u(1,t) = 0; u(x,0) = e^{-x}$.

Solution. This is a heat equation system. We are given $c = 1, L = 1, f(x) = e^{-x}$. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin(n\pi x)$$

where

$$\lambda_n = n\pi$$
$$b_n = 2\int_0^1 f(x)\sin(n\pi x)dx = 2\int_0^1 e^{-x}\sin(n\pi x)dx.$$

The integral in b_n is a repetitive integral, meaning we need to use integration by parts a couple of times to evaluate it. Choose $u = e^{-x}, dv = \sin(n\pi x)dx$:

$$\int e^{-x}\sin(n\pi x)dx = -\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \int \frac{1}{n\pi}e^{-x}\cos(n\pi x)dx$$

Do integration by parts again with $u = e^{-x}, dv = \cos(n\pi x)dx$:

$$\begin{aligned}\int e^{-x}\sin(n\pi x)dx &= -\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \frac{1}{n\pi}\left(\frac{1}{n\pi}e^{-x}\sin(n\pi x) + \int \frac{1}{n\pi}e^{-x}\sin(n\pi x)dx\right) \\ &= -\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \frac{1}{(n\pi)^2}e^{-x}\sin(n\pi x) - \frac{1}{(n\pi)^2}e^{x}\sin(n\pi x)dx.\end{aligned}$$

Notice the integral on the right hand side is the same as the left hand side. Add it to the left hand side:

$$\left(1 + \frac{1}{(n\pi)^2}\right)\int e^{-x}\sin(n\pi x)dx = -\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \frac{1}{(n\pi)^2}e^{-x}\sin(n\pi x)$$

Solve for the integral:

$$\int e^{-x}\sin(n\pi x)dx = \frac{1}{1+\frac{1}{(n\pi)^2}}\left(-\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \frac{1}{(n\pi)^2}e^{-x}\sin(n\pi x)\right)$$

Therefore

$$\begin{aligned}b_n = 2\int_0^1 e^{-x}\sin(n\pi x)dx &= \frac{2}{1+\frac{1}{(n\pi)^2}}\left(-\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \frac{1}{(n\pi)^2}e^{-x}\sin(n\pi x)\right)\Big|_0^1 \\ &= \frac{2(n\pi)^2}{(n\pi)^2+1}\left(-\frac{1}{n\pi}\right)\left(e^{-1}\cos(n\pi) + 1\right) = -\frac{2n\pi}{(n\pi)^2+1}\left(\frac{(-1)^n}{e} + 1\right).\end{aligned}$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{2n\pi}{(n\pi)^2+1}\left(\frac{(-1)^n}{e}+1\right)e^{-(n\pi)^2 t}\sin(n\pi x).$$

□