Math 6B Practice Problems I

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Answers

This page contains answers only. Detailed solutions are on the following pages.

1. $\frac{14}{3}$
2. $\frac{4}{3} - 2\pi$
3. 0
4. (a) 80π
   (b) 80π
5. $\frac{9}{2}$
6. $\frac{32\pi}{3}$
7. (a) convergent, 1
   (b) convergent, 1
   (c) convergent, 1
   (d) convergent, 0
   (e) convergent, 0
   (f) convergent, 0
   (g) divergent
8. (a) divergent
   (b) divergent
   (c) convergent
   (d) divergent
   (e) convergent
   (f) divergent
   (g) convergent
   (h) divergent
   (i) convergent
   (j) convergent
9. See detailed solution

Detailed Solutions

1. Evaluate $\int_C y^2dx + 3xydy$, where C is the boundary of the semiannular region D in the upper half plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$. (You may assume that C is positively oriented.)

   Solution. D is the region $\{(x,y) : 1 \le x^2+y^2 \le 4\} = \{(r,\theta) : 1 \le r \le 2, 0 \le \theta \le \pi\}$. Using Green's Theorem, we have

$$\begin{aligned}\int_C y^2dx + 3xydy &= -\iint_D \left(\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right) dA = \iint_D (3y-2y))\, dA = \iint_D ydA \\ &= \int_0^\pi \int_1^2 r\sin\theta \cdot rdrd\theta = \int_0^\pi \sin\theta d\theta \int_1^2 r^2dr = (-\cos\theta)\Big|_0^\pi \cdot \frac{1}{3}r^3\Big|_1^2 \\ &= \frac{14}{3}\end{aligned}$$

   □

2. Use Green's Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = (\sqrt{x}+y^3, x^2+\sqrt{y})$, and C consists of the arc of the curve $y=\sin x$ from $(0,0)$ to $(\pi,0)$ and the line segment from $(\pi,0)$ to $(0,0)$.

   Solution. Notice that C has negative (clockwise) orientation. We will need to change the sign of our solution at the end. D is the region $\{(x,y) : 0 \le x \le \pi, 0 \le y \le \sin x\}$. Using Green's Theorem, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_D \left(\frac{\partial}{\partial x}(x^2+\sqrt{y}) - \frac{\partial}{\partial y}(\sqrt{x}+y^3)\right) dA = \int_0^\pi \int_0^{\sin x} (2x-3y^2)dydx \\ &= \int_0^\pi \left(2xy - y^3\right)\Big|_0^{\sin x} dx = \int_0^\pi \left(2x\sin x - \sin^3 x\right) dx = \int_0^\pi \left(2x\sin x - \sin x(1-\cos^2 x)\right) dx \\ &= \int_0^\pi \left(2x\sin x - \sin x + \sin x\cos^2 x dx\right) dx = \left(-2x\cos x + 2\sin x + \cos x - \frac{1}{3}\cos^3 x\right)\Big|_0^\pi \\ &= 2\pi - 2 + \frac{2}{3}\end{aligned}$$

   But since C had negative clockwise orientation, we must multiply our solution by negative one:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \frac{4}{3} - 2\pi.$$

   □

3. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = x^2z^2\mathbf{i} + y^2z^2\mathbf{j} + xyz\mathbf{k}$ and S is the part of the paraboloid $z = x^2+y^2$ that lies inside the cylinder $x^2+y^2=4$.

   Solution. We need to find a parametrization $\mathbf{r}(t)$ to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. The closed curve of intersection of the paraboloid and cylinder is the circle $x^2+y^2=4$ at $z=4$, which can be parametrized by

$$\mathbf{r}(t) = (2\cos t, 2\sin t, 4), \quad 0 \le t \le 2\pi.$$

   Then we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt = \int_0^{2\pi} (64\cos^2 t, 64\sin^2 t, 16\cos t\sin t)\cdot(-2\sin t, 2\cos t, 0)dt \\ &= \int_0^{2\pi} (-128\sin t\cos^2 t + 128\sin^2 t\cos t)dt = 128\left(\frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t\right)\Big|_0^{2\pi} = 0.\end{aligned}$$

   □

4. Consider the vector field $\mathbf{F}(x, y, z) = yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}$, where C is circle $x^2 + y^2 = 16, z = 5$ oriented counterclockwise when viewed from above.

(a) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by finding an appropriate parametrization vector $\mathbf{r}(t)$.

Solution. The parametrization for the circle $x^2 + y^2 = 16, z = 5$ is given by

$$\mathbf{r}(t) = (4\cos t, 4\sin t, 5), \quad 0 \le t \le 2\pi.$$

Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^{2\pi} (20\sin t, 40\cos t, e^{16\cos t \sin t}) \cdot (-4\sin t, 4\cos t, 0)dt \\
&= \int_0^{2\pi} (-80\sin^2 t + 160\cos^2 t)dt = \int_0^{2\pi} (-40(1-\cos 2t) + 80(1+\cos 2t))dt \\
&= \int_0^{2\pi} (40 + 120\cos 2t)dt = (40t + 60\sin 2t)\Big|_0^{2\pi} = 80\pi.
\end{aligned}$$

□

(b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem, and verify it is equal to your solution in part (a).

Solution. We need to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$. The normal to the surface is the normal to $z = 5$, which is $\mathbf{n} = \mathbf{k}$. The curl of F is given by

$$\begin{aligned}
\text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & e^{xy} \end{vmatrix} = \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & e^{xy} \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ yz & e^{xy} \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yz & 2xz \end{vmatrix} \\
&= \mathbf{i}(xe^{xy} - 2x) - \mathbf{j}(ye^{xy} - y) + \mathbf{k}(2z - z).
\end{aligned}$$

Then curl $\mathbf{F} \cdot \mathbf{n} = z$, but on our surface $z = 5$, hence

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D z dA = \iint_D 5dA = 5 \cdot \pi \cdot 4^2 = 80\pi.$$

□

5. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x, y, z) = 3x\mathbf{i} + xy\mathbf{j} + 2xz\mathbf{k}$ where E is the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
Note: to verify the theorem is true you need to show that $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV$; that is, you need to calculate both integrals and show they are equal.

Solution. We will evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ on each of the different faces of the cube then take their sum.
On $x = 0$, $\mathbf{n} = -\mathbf{i}$, $D = \{(y, z) : 0 \le y \le 1, 0 \le z \le 1\}$ and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D -3x dA = \iint_D 0 dA = 0.$$

On $x = 1$, $\mathbf{n} = \mathbf{i}$, $D = \{(y, z) : 0 \le y \le 1, 0 \le z \le 1\}$ and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D 3x dA = \iint_D 3 dA = 3.$$

On $y = 0$, $\mathbf{n} = -\mathbf{j}$, $D = \{(x, z) : 0 \le x \le 1, 0 \le z \le 1\}$ and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D -xy dA = \iint_D 0 dA = 0.$$

On $y=0$, $\mathbf{n}=\mathbf{j}$, $D=\{(x,z):0\le x\le 1, 0\le z\le 1\}$ and

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D xy dA=\iint_D x dA=\int_0^1\int_0^1 x dx dz=\frac{1}{2}.$$

On $z=0$, $\mathbf{n}=-\mathbf{k}$, $D=\{(x,y):0\le x\le 1, 0\le z\le 1\}$ and

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D -2xz dA=\iint_D 0 dA=0.$$

On $z=1$, $\mathbf{n}=\mathbf{k}$, $D=\{(x,y):0\le x\le 1, 0\le z\le 1\}$ and

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D 2xz dA=\iint_D 2x dA=\int_0^1\int_0^1 2x dx dy=1.$$

Therefore

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=0+3+0+\frac{1}{2}+0+1=\frac{9}{2}.$$

Now we will evaluate $\iiint_E \text{div } \mathbf{F} dV$ where $E=\{(x,y,z):0\le x\le 1, 0\le y\le 1, 0\le z\le 1\}$. The divergence of F is given by

$$\text{div } \mathbf{F}=\frac{\partial}{\partial x}(3x)+\frac{\partial}{\partial y}(xy)+\frac{\partial}{\partial z}(2xz)=3+x+2x=3x+3.$$

Therefore

$$\iiint_E \text{div } \mathbf{F} dV=\iiint_E (3x+3) dV=\int_0^1\int_0^1\int_0^1(3x+3)dx dy dz=\left(\frac{3}{2}x^2+3x\right)\Big|_0^1=\frac{3}{2}+3=\frac{9}{2}.$$

□

6. Use the Divergence Theorem to calculate the surface integral $\iint \mathbf{F}\cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S where $\mathbf{F}(x,y,z)=(\cos z+xy^2)\mathbf{i}+xe^{-z}\mathbf{j}+(\sin y+x^2z)\mathbf{k}$, and S is the surface of the solid bounded by the paraboloid $z=x^2+y^2$ and the plane $z=4$.

   Solution. We will evaluate $\iiint_E \text{div } \mathbf{F} dV$ where $E=\{(x,y,z):x^2+y^2\le z\le 4\}=\{(r,\theta,z):0\le r\le 2, 0\le\theta\le 2\pi, r^2\le z\le 4\}$. The divergence of F is given by

$$\text{div } \mathbf{F}=\frac{\partial}{\partial x}(\cos z+xy^2)+\frac{\partial}{\partial y}(xe^{-z})+\frac{\partial}{\partial z}(\sin y+x^2z)=y^2+x^2.$$

   Then

$$\begin{aligned}\iiint_E \text{div } \mathbf{F} dV&=\iiint_E (x^2+y^2)dV=\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cdot r dz dr d\theta=2\pi\int_0^2\int_0^{2\pi} r^3 dz dr=2\pi\int_0^2 r^3 z\Big|_{z=r^2}^{z=4}dr\\&=2\pi\int_0^2 r^3(4-r^2)dr=2\pi\int_0^2(4r^3-r^5)dr=2\pi\left(r^4-\frac{1}{6}r^6\right)\Big|_0^2=2\pi\left(16-\frac{64}{6}\right)\\&=\frac{32\pi}{3}.\end{aligned}$$

□

7. Determine whether the sequence converges or diverges. If it converges, find the limit.

   (a) $a_n=e^{1/n}$

Solution. Since e^x is a continuous function, we can pass the limit through the function:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{1/n} = \exp\left(\lim_{n\to\infty} \frac{1}{n}\right) = e^0 = 1.$$

Thus a_n converges to 1. □

(b) $a_n = n \sin\left(\frac{1}{n}\right)$

Solution. Let $f(x) = x\sin\left(\frac{1}{x}\right)$. We will evaluate the limit as $x \to \infty$ using L'Hôpital's rule:

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot -\frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos\left(\lim_{x\to\infty}\frac{1}{x}\right) = \cos 0 = 1.$$

Notice we used the fact that cosine is continuous to pass the limit through the function. Thus a_n converges to 1. □

(c) $a_n = 1 - (0.2)^n$

Solution.

$$\lim_{n\to\infty}(1 - (0.2)^n) = 1 - \lim_{n\to\infty}\left(\frac{1}{5}\right)^n = 1 - \lim_{n\to\infty}\frac{1}{5^n} = 1 - 0 = 1.$$

Thus a_n converges to 1. □

(d) $a_n = n^2 e^{-n}$

Solution. Let $f(x) = x^2 e^{-x}$. We will evaluate the limit as $x \to \infty$ using L'Hôpital's rule:

$$\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0.$$

Thus a_n converges to 0. □

(e) $a_n = \dfrac{(-1)^{n-1}n}{n^2+1}$

Solution. We will check to see if $\lim_{n\to\infty}|a_n|$ converges:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n-1}n}{n^2+1}\right| = \lim_{n\to\infty}\frac{n}{n^2+1} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0.$$

Since $|a_n|$ converges to 0, by a theorem we have a_n also converges to 0. □

(f) $a_n = \dfrac{\cos^2 n}{2^n}$

Solution. We will use Squeeze Theorem. Since $0 \le \cos^2 n \le 1$, then

$$0 \le \frac{\cos^2 n}{2^n} \le \frac{1}{2^n}.$$

Take the limit on each side as $n \to \infty$:

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty}\frac{\cos^2 n}{2^n} \le \lim_{n\to\infty}\frac{1}{2^n}$$

$$0 \le \lim_{n\to\infty}\frac{\cos^2 n}{2^n} \le 0.$$

Thus a_n converges to 0 by Squeeze Theorem. □

(g) $a_n = \dfrac{n^n}{n!}$

Solution. Write out a general term of the sequence:

$$a_n = \frac{n \cdot n \cdot n \cdots n \cdot n}{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n} = \frac{n}{1}\left(\frac{n}{2} \cdot \frac{n}{3} \cdots \frac{n}{n-1} \cdot \frac{n}{n}\right)$$

The rightmost term is $\frac{n}{n} = 1$. Since $n - 1 < n$, then $\frac{n}{n-1} \geq 1$. In fact, each fraction in the parenthesis will be greater than 1 for large enough n:

$$a_n = \frac{n}{1}\left(\frac{n}{2} \cdot \frac{n}{3} \cdots \frac{n}{n-1} \cdot \frac{n}{n}\right) \geq n\,(1 \cdot 1 \cdots 1 \cdot 1) = n.$$

Therefore $a_n \geq n$, and since $\lim_{n\to\infty} n$ diverges, so does a_n. □

8. Determine whether the series is convergent or divergent. State what test(s) you used to come to your conclusion.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$

Solution. We can rewrite the sum:

$$\sum_{n=1}^{\infty} \frac{1+3^n}{2^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2^n} + \frac{3^n}{2^n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n.$$

Both of these series are geometric series. The series on the left with $r = \frac{1}{2}$ will converge, however the series on the right with $r = \frac{3}{2} > 1$ will diverge. Hence the series is divergent (by geometric series). □

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

Solution. Let $f(x) = \dfrac{e^x}{x^2}$. We will evaluate the limit as $x \to \infty$ using L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

The limit diverges, hence the series diverges by the Divergence Test. □

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

Solution. Let $f(x) = xe^{-x}$. f is continuous, positive, decreasing ($f' < 0$), so we can apply the integral test:

$$\int_1^{\infty} xe^{-x}dx = \left(-xe^{-x} - e^{-x}\right)\Big|_1^{\infty} = -\lim_{x\to\infty} \frac{x+1}{e^x} - \left(-e^{-1} - e^{-1}\right)$$

The limit converges using L'Hôpital's rule, therefore the integral converges. Since the integral converges, the series also converges by the Integral Test. □

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{0.85}}$

Solution. We can rewrite the sum:

$$\sum_{n=1}^{\infty} \frac{2}{n^{0.85}} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{0.85}}.$$

Since $p = 0.85 < 1$, this series diverges by the p-Series Test. □

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{10^n}$

Solution. Since $\sin n \leq 1$, then $1 + \sin n \leq 2$, and we have

$$\frac{1+\sin n}{10^n} \leq \frac{2}{10^n}.$$

The series

$$\sum_{n=1}^{\infty} \frac{2}{10^n} = 2 \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$$

is a convergent geometric series with $r = \frac{1}{10} < 1$. Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{10^n}$ converges by the Comparison Test. □

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

Solution. Since $n + 1 > n$, then

$$\frac{n+1}{n\sqrt{n}} > \frac{n}{n\sqrt{n}} \quad \Rightarrow \quad \frac{n+1}{n\sqrt{n}} > \frac{1}{\sqrt{n}}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent p-series with $p = \frac{1}{2} < 1$. Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$ is divergent by the Comparison Test. □

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{10^n}$

Proof. We will use the Alternating Series Test. Let $a_n = \dfrac{n}{10^n}$. We need to show that a_n forms a decreasing sequence. Let $f(x) = \dfrac{x}{10^x}$, we will show that $f' < 0$:

$$f'(x) = \frac{10^x(x)' - x(10^x)'}{10^{2x}} = \frac{10^x - x\ln 10(10)^x}{10^{2x}} = \frac{10^x(1 - x\ln 10)}{10^{2x}} = \frac{1 - x\ln 10}{10^x} < 0$$

for $x > 1$. Thus a_n is decreasing.
Now we need to show a_n converges to 0. We will do this by using L'Hôpital's Rule to show that $f(x)$ converges to 0 as $x \to \infty$:

$$\lim_{x\to\infty} \frac{x}{10^x} = \lim_{x\to\infty} \frac{1}{10^x \ln 10} = 0.$$

Thus a_n converges to 0.
Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{10^n}$ converges by the Alternating Series Test. □

(h) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{\pi}{n}\right)$

Solution. Since cosine is continuous,

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \cos 0 = 1.$$

Since the limit is not equal to zero, then the series diverges by the Divergence Test. □

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$

Solution. Let $a_n = \dfrac{(-10)^n}{n!}$. We will use the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-10)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-10)^n}\right| = \lim_{n\to\infty} \left|\frac{-10}{n+1}\right| = \lim_{n\to\infty} \frac{10}{n+1} = 0 < 1,$$

therefore the series is convergent by the Ratio Test. □

(j) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

Solution. Let $a_n = \left(\dfrac{n^2+1}{2n^2+1}\right)^n$. We will use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1,$$

therefore the series is convergent by the Root Test. □

9. Use the Integral test to prove that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Solution. Let $f(x) = \dfrac{1}{x}$. f is continuous, positive, decreasing on $[1, \infty)$, so we can apply the Integral Test. The integral

$$\int_1^{\infty} \frac{1}{x} = \ln x \Big|_1^{\infty} = \lim_{x\to\infty} x - \ln 1$$

diverges, hence by the Integral Test the harmonic series diverges. □