Math 6B: Sequences "Quiz" Solutions

April 21, 2016

Determine if the following sequences converge or diverge. If the sequence converges, find its limit. Be sure to explain what theorems you used to reach your conclusion.

1. $a_n = \dfrac{3+5n^2}{n^2+n}$

   Solution. a_n is convergent and converges to 5 since since

$$\lim_{n\to\infty} \frac{3+5n^2}{n^2+n} = \lim_{n\to\infty} \frac{\frac{3}{n^2}+5}{1+\frac{1}{n}} = 5.$$

□

2. $a_n = \dfrac{n\cos n^2}{n^2+100}$

   Solution. $-1 \le \cos n^2 \le 1$, therefore

$$-\frac{n}{n^2+100} \le \frac{n\cos n^2}{n^2+100} \le \frac{n}{n^2+100}.$$

   Taking the limit on each side of the inequality, we have

$$\lim_{n\to\infty} -\frac{n}{n^2+100} \le \lim_{n\to\infty} \frac{n\cos n^2}{n^2+100} \le \lim_{n\to\infty} \frac{n}{n^2+100}$$

$$\Rightarrow \quad 0 \le \lim_{n\to\infty} \frac{n\cos n^2}{n^2+100} \le 0$$

   By Squeeze Theorem, a_n converges and $\lim_{n\to\infty} a_n = 0$. □

3. $a_n = \dfrac{(-1)^n n^4}{n^3+2n^2+1}$

   Solution.

$$\lim_{n\to\infty} \frac{(-1)^n n^4}{n^3+2n^2+1} = \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \frac{n^4}{n^3+2n^2+1}$$

   The right limit diverges since the numerator has higher degree than the denominator, therefore a_n is divergent. □

4. $a_n = \ln(2n^2+1) - \ln(n^2+1)$

   Proof. The natural log function $\ln x$ is continuous, therefore

$$\lim_{n\to\infty} \left(\ln(2n^2+1) - \ln(n^2+1)\right) = \lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right) = \ln\left(\lim_{n\to\infty} \frac{2n^2+1}{n^2+1}\right) = \ln 2.$$

   Thus a_n is convergent and converges to $\ln 2$. □

5. $a_n = \dfrac{2^n}{n!}$

Proof. Write out a_n and rearrange some terms:

$$a_n = \frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2}{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n} = \frac{2}{n} \cdot \frac{2}{1} \cdot \frac{2}{2} \left(\frac{2}{3} \cdot \frac{2}{4} \cdots \frac{2}{n-1} \right)$$

Notice that a_n is positive and that $\frac{2}{3} \leq 1$, $\frac{2}{4} \leq 1$, ... $\frac{2}{n-1} \leq 1$. Therefore

$$0 \leq a_n \leq \frac{2}{n} \cdot \frac{2}{1} \cdot \frac{2}{2} \cdot 1 \cdot 1 \cdots 1$$

$$0 \leq a_n \leq \frac{4}{n}$$

Take the limit on each side of the inequality:

$$\lim_{n \to \infty} 0 \leq \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} \frac{4}{n}$$

$$0 \leq \lim_{n \to \infty} a_n \leq 0.$$

By Squeeze Theorem, a_n converges and $\lim_{n \to \infty} a_n = 0$. □

6. $a_n = \dfrac{n!}{2^n}$

Proof. This will be similar to the previous problem. Write out a_n and rearrange some terms:

$$a_n = \frac{n!}{2^n} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n}{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2} = \frac{n}{2} \cdot \frac{1}{2} \cdot \frac{2}{2} \left(\frac{3}{2} \cdot \frac{4}{2} \cdots \frac{n-1}{2} \right)$$

Notice that $\frac{3}{2} \geq 1, \frac{4}{2} \geq 1, ... \frac{n-1}{2} \geq 1$. Therefore

$$a_n \geq \frac{n}{2} \cdot \frac{1}{2} \cdot \frac{2}{2} \cdot 1 \cdot 1 \cdots 1 \quad \Rightarrow \quad a_n \geq \frac{n}{4}.$$

Since $\lim_{n \to \infty} \frac{n}{4} = \infty$, then a_n diverges. □