Math 6B: Series "Quiz" Solutions

April 21, 2016

Determine if the following series converge or diverge. State what test you used to come to your conclusion.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.2}}$

   Proof. Since $\arctan n \leq \frac{\pi}{2}$, we have
   $$\frac{\arctan n}{n^{1.2}} \leq \frac{\frac{\pi}{2}}{n^{1.2}}.$$
   The series
   $$\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.2}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$$
   converges by the p-Series test since $p = 1.2 > 1$. Therefore our series converges by the Comparison Test. □

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

   Proof. Consider the function $f(x) = \dfrac{1}{x \ln x}$ on the interval $[2, \infty)$. Using the substitution $u = \ln x, du = \frac{1}{x} dx$, we see that the integral diverges:
   $$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_{\ln 2}^{\infty} \frac{1}{u} du = \ln u \Big|_{\ln 2}^{\infty} = \infty$$
   Therefore the series diverges by the Integral Test. □

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

   Proof. The series diverges by the p-Series test since $p = \frac{1}{5} \leq 1$. □

4. $\displaystyle\sum_{n=1}^{\infty} \arctan n$

   Proof. The series diverges by the Divergence Test since
   $$\lim_{n \to \infty} \arctan n = \frac{\pi}{2} \neq 0.$$
   □

5. $3 + 2 + \dfrac{4}{3} + \dfrac{8}{9} + ...$

   Proof. This is a geometric series since each term is a multiple of the last. The ratio is $r = \frac{2}{3}$ (divide the second term by the first term, the third term by the second term, etc.). The sum can therefore be written as
   $$\sum_{n=1}^{\infty} 3 \left(\frac{2}{3}\right)^{n-1}$$
   Since $|r| < 1$, the series converges, and it converges to $\dfrac{3}{1 - \frac{2}{3}} = \dfrac{3}{\frac{1}{3}} = 9$. □

6. $\sum_{n=1}^{\infty} \frac{1}{n^5}$

Proof. The sequence converges by the p-Series test since $p = 5 > 1$. $\square$

7. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Proof. Since $n > 1$, then $n^2 > 1$. Therefore

$$n^2 + 1 \leq n^2 + n^2 \quad \Rightarrow \quad \sqrt{n^2+1} \leq \sqrt{n^2+n^2}.$$

Thus

$$\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+n^2}}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n^2}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2}} = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (it is a harmonic series), by the Comparison Test our series is divergent. $\square$