Math 33A − Week 5

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Name: ____

1. Let $T(\mathbf{x}) = \begin{pmatrix} 1 & -3 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix} \mathbf{x}$.

   (a) Find $\text{im}(T)$ and $\ker(T)$.

   (b) Verify the rank-nullity theorem for this transformation.

   (c) Are there any vectors $\mathbf{y} \in \mathbb{R}^2$ such that $\mathbf{y} \notin \text{im}(T)$?

2. Recall that $T(\mathbf{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}$ is the transformation that projects $\mathbf{x} = (x, y)$ onto the y-axis.

   (a) Without solving explicitly, what do you expect $\text{im}(T)$ and $\ker(T)$ to be? (You don't need to use set notation here, you can write a short description about what you think they will be.)

   (b) Solve $\text{im}(T)$ and $\ker(T)$ explicitly. Do these match your intuition in (a)?

(c) Verify the rank-nullity theorem for this transformation.

(d) Find a vector $\mathbf{y}$ such that $\mathbf{y} \notin \text{im}(T)$.

(e) Is $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ invertible?

3. Recall that $T(\mathbf{x}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x}$ is the transformation that rotates $\mathbf{x}$ counterclockwise by $\frac{\pi}{4}$ radians and stretches the rotated vector by a factor of $\sqrt{2}$.

   (a) Without solving explicitly, what do you expect $\text{im}(T)$ and $\ker(T)$ to be? (You don't need to use set notation here, you can write a short description about what you think they will be.)

   (b) Solve $\text{im}(T)$ and $\ker(T)$ explicitly. Do these match your intuition in (a)?

   (c) Verify the rank-nullity theorem for this transformation.

   (d) Are there any vectors $\mathbf{y} \in \mathbb{R}^2$ such that $\mathbf{y} \notin \text{im}(T)$?

   (e) Is $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ invertible?

4. The invertibility of a square matrix is related to its rank, nullity, kernel, and image. The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in Exercise 2 is not invertible, but $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ in Exercise 3 is invertible. Fill in the following statements below (you can use Exercises 2 and 3 for reference):

   (a) If A is an $n \times n$ matrix and $\text{rank}(A) =$ ______, then $\text{im}(A) = \mathbb{R}^n$.

   (b) If A is an $n \times n$ matrix and $\text{rank}(A) = n$, then $\text{nullity}(A) =$ ______.

   (c) If A is an $n \times n$ matrix and $\text{nullity}(A) =$ ______, then $\ker(A) = \{\mathbf{0}\}$.

   (d) If A is an $n \times n$ matrix and $\ker(A) =$ ______, then A is invertible.

   (e) If A is an $n \times n$ matrix and $\text{rank}(A) \neq$ ______, A is not invertible.

   (f) If A is an $n \times n$ matrix and $\text{nullity}(A) \neq$ ______, A is not invertible.

   (g) If A is an $n \times n$ matrix and $\ker(A) \neq \{\mathbf{0}\}$, A is __________.