Math 33B Worksheet 7 (Phase Plane for Linear Systems)

Name: ______________________________ Score: __________

Circle the name of your TA: Ziheng Nicholas Victoria

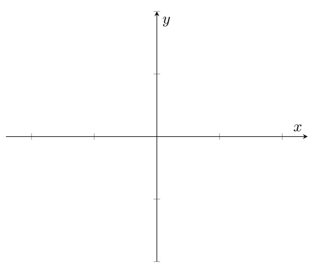
Circle the day of your discussion: Tuesday Thursday

1. The linear system

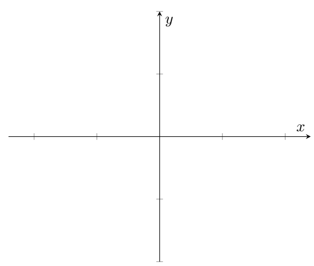
$$x' = ax + by$$
$$y' = cx + dy$$

can be written in matrix form $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We can use the eigenvalues and eigenvectors of A to sketch the phase plane/portrait for the system. Below are the possible behaviors for linear systems with nonzero eigenvalues, sketch a plausible phase portrait for each behavior.

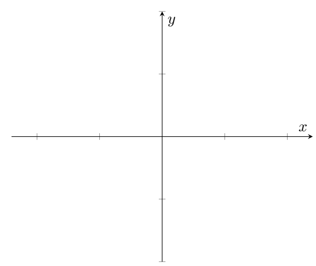
(i) Real, distinct eigenvalues ($\lambda_1 \neq \lambda_2$)



unstable node
($\lambda_1 > \lambda_2 > 0$)

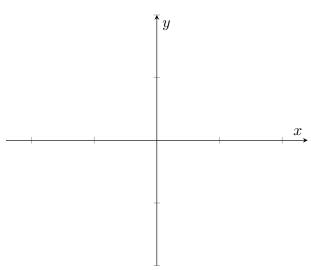


saddle
($\lambda_1 > 0 > \lambda_2$)

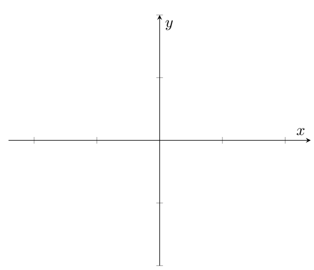


stable node
($\lambda_1 < \lambda_2 < 0$)

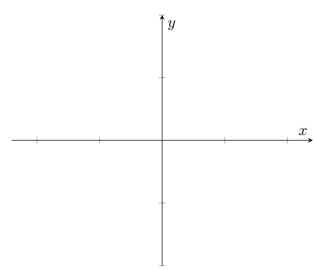
(ii) Complex eigenvalues ($\lambda = \alpha \pm \beta i$)



center
($\alpha = 0$)



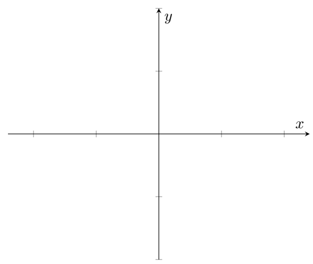
stable spiral
($\alpha < 0$)



unstable spiral
($\alpha > 0$)

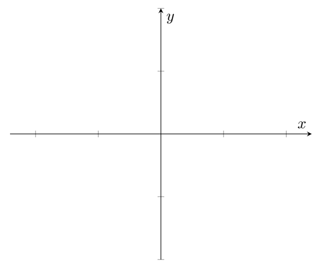
Question: How do we determine whether the center or spiral is clockwise or counter-clockwise?

(iii) Real repeated eigenvalues



star node

Occurs when $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$



degenerate node

Occurs when $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$

Question: When are each of these nodes stable or unstable?

2. Consider the linear system

$$x' = x + y$$
$$y' = 4x - 2y.$$

(a) Rewrite this system in matrix form $\mathbf{x}' = A\mathbf{x}$. Find the eigenvalues of A and use these to classify the behavior and stability of the system.

(b) Find the eigenvectors of A and sketch the phase portrait of the system.

3. Consider the linear system

$$x' = 4x - y$$
$$y' = 2x + y.$$

(a) Rewrite this system in matrix form $\mathbf{x}' = A\mathbf{x}$. Find the eigenvalues of A and use these to classify the behavior and stability of the system.

(b) Find the eigenvectors of A and sketch the phase portrait of the system.

4. Consider the linear system

$$x' = x - y$$
$$y' = x + y.$$

(a) Rewrite this system in matrix form $\mathbf{x}' = A\mathbf{x}$. Find the eigenvalues of A and use these to classify the behavior and stability of the system.

(b) The eigenvalues in (a) are complex with nonzero real part. Come up with a way to determine whether the spiral is clockwise or counterclockwise, then sketch the phase portrait of the system.

5. **Simple Harmonic Oscillator.** The vibrations of a mass hanging from a linear spring are governed by the linear differential equation

$$mx'' + kx = 0$$

where m is the mass, k is the spring constant, and x is the displacement of the mass from the equilibrium. We can rewrite this second order differential equation as the first order system

$$\begin{aligned} x' &= v \\ v' &= -\omega^2 x \end{aligned}$$

where $\omega^2 = k/m$. Plot the phase portrait for this system. What do these closed orbits represent?